Examination Papers

MA 414	Duration	Max. Marks 10
Quiz 1	40 minutes	Weightage 10 %

(1) Show that if a regular polygon of p sides, where p is a prime number, is constructible by ruler and compass, then p is a Fermat prime. [2]

(2) Let F be a field and x be an indeterminate. Find all the intermediate fields of $F(x)/F(x^p)$ where p is a prime number. [2]

(3) Let K/F be a field extension of degree n. Let $a \in K$ and $\mu_a : K \to K$ be the linear map $\mu_a(x) = ax$ for all $x \in K$. Show that a is the root of the characteristic polynomial of μ_a. Use this fact to find the irreducible polynomial over $\mathbb{Q}$ of $\alpha = 1 + \beta + \beta^2$ where $\beta = \sqrt[3]{2}$. [3]

(4) Find the irreducible polynomial of $\cos 2\pi/11$ over $\mathbb{Q}$ and show that it is not possible to construct a regular polygon of 11 sides by ruler and compass. [3]

MA 414	Duration	Max. Marks 10
Quiz 2	40 minutes	Weightage 10 %

(1) Find all the $\mathbb{F}_q$-automorphisms of $\mathbb{F}_{q^n}$. [2].

(2) Find the number of monic irreducible polynomials of degree 4 over $\mathbb{F}_2$ by using Gauss's formula. List these polynomials. [2]

(3) Let $\omega = e^{2\pi i/3}$. Show that $\omega\sqrt{5}$ is a primitive element of $\mathbb{Q}(\omega, \sqrt{5})$ over $\mathbb{Q}$. Find $\operatorname{irr}(\omega\sqrt{5}, \mathbb{Q})$. [3]

(4) Let p be a prime number and u, v, w be indeterminates over the finite field $\mathbb{F}_p$. Show that the field extension $\mathbb{F}_p(u, v, w)/\mathbb{F}_p(u^p, v^p, w^p)$ has no primitive element. List infinitely many subfields of the field extension $\mathbb{F}_p(u, v, w)/\mathbb{F}_p(u^p, v^p, w^p)$. [3]

MA 414	Duration	Max. Marks 10
Quiz 3	**40 minutes**	**Weightage 10 %**

(1) Find the Galois group of $f(x) = x^3 - 3x + 1$ over $\mathbb{Q}$. [2]

(2) Let p be a prime number and $q = p^n$ for some natural number n. Show that $G(\mathbb{F}_q/\mathbb{F}_p)$ is a cyclic group. [2]

(3) Let G be a finite group of automorphisms of a field E. Show that E is a Galois extension of the subfield

$$E^G = \{a \in E \mid g(a) = a \text{ for all } g \in G\}.$$

Show that $G(E/E^G) = G$. [3]

(4) Let t be an indeterminate and $\omega = e^{2\pi i/3}$. Let $E = \mathbb{C}(t)$ and $F = \mathbb{C}(t^3 + t^{-3})$. Show that the maps σ, τ defined by $\sigma(t) = \omega t$ and $\tau(t) = 1/t$ are F-automorphisms of E. Describe all the automorphims in $G(E/F)$. [3]

MA 414	Duration	Max. Marks 10
Quiz 4	**40 minutes**	**Weightage 10 %**

(1) Let p be a prime number and $n \in \mathbb{N}$. Show that if $p \nmid n$ then

$$\Phi_{pn}(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)}.$$

[2]

(2) Show that $\mathbb{Q}\left(\sqrt{(-1)^{\binom{p}{2}}p}\right)$ is the unique quadratic extension of $\mathbb{Q}$ in $\mathbb{Q}(\zeta_p)$. [3]

(3) Let $z = \zeta_{11}$. Find the polynomial $\operatorname{irr}(z + z^3 + z^4 + z^5 + z^9, \mathbb{Q})$. [2]

(4) Write $G = G(\mathbb{Q}(\zeta_{15})/\mathbb{Q})$ as a product of two cyclic subgroups. Find all square free integers n such that $\mathbb{Q}(\sqrt{n})$ are fixed fields of subgroups of G. [3]

MA 414	Duration	Max. Marks 10
Quiz 5	40 minutes	Weightage 10 %

You may use the fact that the resolvent cubic of $x^4 + bx^2 + cx + d$ is $x^3 - bx^2 - 4dx - c^2 + 4bd$.

(1) Let $f(x) \in F[x]$ be an irreducible quartic where char $F \neq 2, 3$. Suppose that it has exactly two real roots. Show that $G_f = D_4$ or S_4. [3]

(2) Find the Galois group of $x^4 + 1$ over $\mathbb{Q}$. [3]

(3) Let $h(x) \in \mathbb{Q}[x]$ be a monic polynomial of degree n. Show that G_h is a transitive subgroup of S_n if and only if $h(x)$ is irreducible in $\mathbb{Q}[x]$. [4]

MA 414	Duration	Max. Marks 10
Quiz 6	40 minutes	Weightage 10 %

(1) Let x_1, x_2, x_3 be indeterminates and let s_1, s_2, s_3 be the elementary symmetric polynomials in x_1, x_2, x_3. Show that $E = \mathbb{Q}(x_1, x_2, x_3)$ is not a radical extension of $F = \mathbb{Q}(s_1, s_2, s_3)$. What is $G(E/F)$? [3]

(2) Find the Galois group of $p(x) = x^5 - 6x + 3$ over $\mathbb{Q}$. Is $p(x)$ solvable by radicals over $\mathbb{Q}$? [3]

(3) Find the Galois group of $q(x) = x^3 - 3x + 1$ over $\mathbb{Q}$. Is the splitting field of $q(x)$ over $\mathbb{Q}$ a radical extension of $\mathbb{Q}$? [4]

MA 414	Duration	Max. Marks 30
Mid-Sem	2 hrs	Weightage 30 %

Instructions: (1) E, F, K will denote fields. (2) p denotes a prime number.

(1) Let char $F = p > 0$. Let $f(x) \in F[x]$ be an irreducible separable polynomial of degree d with only one root. Find $f(x)$. [3]

(2) Let $k = \mathbb{F}_p$ and $k(x)$ denote the field of rational functions in the variable x with coefficients in k. Put $f(x) = x^p - a^{p-1}x$ where $a \in k^\times$. Show that the roots of $f(x)$ in the algebraic closure $\overline{k}$ of k form an additive subgroup of $\overline{k}$. Find the elements of this group. [3]

(3) Show that $x^{p^n} - a \in F[x]$ where char $F = p > 0$ is either irreducible or $a \in F^p$. [3]

(4) Describe and justify a ruler-compass construction of a regular pentagon. [3]

(5) Let E/F be a finite algebraic extension of finite fields. Show that the set $E^\times$ of nonzero elements of E is a cyclic group. [3]

(6) Let E/F be a finite algebraic extension and $E = F(a)$ for some $a \in E$. Show that the number of intermediate subfields of E/F is finite. [3]

(7) Let $\sigma : \mathbb{R} \to \mathbb{R}$ be an automorphism. Show that if $a > 0$ then $\sigma(a) > 0$. Find all automorphisms of $\mathbb{R}$. [4]

(8) Let M be an $n \times n$ matrix with complex entries. Show that M is nilpotent if and only of the trace of M^i is zero for all $i = 1, 2, \ldots$. [4]

(9) Find the number of irreducible factors of $f(x) = x^{3^{15}} - x$ in $\mathbb{F}_3$. Let E denote a splitting field of $f(x)$ over $\mathbb{F}_3$. Draw a diagram of subfields of $E/\mathbb{F}_3$. [4]

MA 414	Duration	Max. Marks 30
End-Sem	2 hrs	Weightage 30 %

Instructions

(1) E, F, K will denote fields. (2) p denotes a prime number and $q = p^n$.
(3) Justify all statements. (4) Each question carries 3 marks.

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(1) Let $E/\mathbb{F}_q$ be a finite extension. Show that $N_{E/\mathbb{F}_q} : E^\times \to \mathbb{F}_q^\times$ is surjective.

(2) Let F be a field of characteristic p. Let E/F be a cyclic extension of degree p. Show that E is a spliting field of $f(x) = x^p - x - a$ for some $a \in F$.

(3) Find the Galois group of $f(x) = x^4 + 5x + 5$ over $\mathbb{Q}$.

(4) Let $f(x)$ be an irreducible quintic over $\mathbb{Q}$ with exactly two non-real roots. Find the Galois group of $f(x)$ over $\mathbb{Q}$.

(5) Find the cyclotomic polynomial $\Phi_{100}(x)$ and its Galois group over $\mathbb{Q}$.

(6) Show that a finite group G is solvable if and only if $G^{(s)} = \{1\}$ for some s.

(7) Let ζ be a primitive 7^{th} root of unity. Find the Galois group of the irreducible polynomial of $\zeta + \zeta^5$ over $\mathbb{Q}$.

(8) Find the discriminant of $\Phi_p(x)$.

(9) Show that a regular polygon of p sides is constructible by ruler and compass if and only if p is a Fermat prime.

(10) Let F be a field of characteristic $\neq 2$. Consider the quartic polynomial $f(x) = x^4 + bx^2 + cx + d$. Let r_1, r_2, r_3 and r_4 be the roots of $f(x)$ in a splitting field E of $f(x)$ over F. The resolvent cubic for $f(x)$ having roots

$$t_1 = r_1r_2 + r_3r_4,\ t_2 = r_1r_3 + r_2r_4,\ t_3 = r_2r_3 + r_1r_4,$$

is $g(x) = x^3 - bx^2 - 4dx - c^2 + 4bd$. Let $K = F(t_1, t_2, t_3)$. Find the Galois group $G(K/F)$.