

LINEAR PROGRAMMING

Introduction to Lecture

Teacher (T) – Welcome to this interactive session to understand some principles in Operations Research

Student (S) – I thank you Sir for inviting me to this interactive session. I hope you are aware that I do not have any background in this subject. I am a little worried that I may struggle to cope up with this very tough subject

T – Well, I do not think this subject is as difficult as you think. We will make it interesting and enjoyable for you. I hope that at the end you will thoroughly enjoy these exercises.

S- Thank you sir, I look forward to your guidance

T – Let us begin with a simple problem

T – Consider a person making tables and chairs. The person has 15 units of steel. Each table requires 3 units of steel and each chair requires 2 units of steel. The person has 15 hours to spend on making the tables and chairs. Each table requires 2 hours and each chair requires 3 hours to make. The person earns Rs 80 per table and Rs 70 per chair. How many tables and chairs does the person make?

S – This is difficult for me.

T – Let me see how you approach this problem

S – Let me assume that the person makes 10 tables. Since the profit per table is 80, can i assume that the profit is 800?

T – Yes. You can assume so. You have assumed that the profit is proportional to the units sold. In fact you can assume that if you make 10 tables and 10 chairs, the total profit is 1500.

S – I was going to ask that next.

T – You said that you wish to make 10 tables. Can you make 10 tables with the available resources?

S – Ten tables require 30 units of steel and 20 hours. Oh! We do not have the required resources.

T – You can make as many tables and chairs as the resources allow you to do so.

S – Based on the amount of steel available, i can make 5 tables and make Rs 400

T – It is true that you can make 5 tables using the 15 units of steel. Should you not check whether you have the time to make 5 tables?

S – Oh yes. I forgot. Five tables require 10 hours and luckily i have 15 hours. I can therefore make 5 tables and get a revenue of Rs 400. I understand that I can only make as many tables or chairs as the resources permit and that i should have all the resources to make a certain number of tables or chairs.

T – Do you want to decide on the number of tables based on the amount of time available?

S – I have 15 hours and each table requires 2 hours. I can therefore make 7.5 tables and get revenue of Rs 1200. How can I make 7.5 tables?

T – At present you can resolve this issue by considering 7 tables.

*S – Seven tables require 21 units of material but i have only 15. Therefore i can make a maximum of 5 tables, which i have already found out. I will therefore conclude that i can make **5 tables** and earn revenue of **Rs 400**.*

T – In the answer that you provided, you did not make any chair. Don't you want to consider making chairs?

S – Yes. Let me look at the problem considering the chairs. With 15 units of material, I can make 7.5 chairs. I assume that I can definitely make 7. The time required is 21 which is more than what i have. Based on the time available, I can make 5 chairs and I have 10 units of material needed. In fact I have 15. Therefore I can make 5 chairs. This is also a good solution.

T – Should you not see how much money you make?

S – Yes. I would make Rs 350. Is this a good solution?

T – Now we have two solutions that you have given. Both make 5 units of a product – different products. The revenues are 400 and 350. You should choose the solution that gives you higher revenue.

S – My purpose or objective is to have high revenue.

T – Your objective is to maximize the revenue.

S – I will choose the solution that gives higher revenue. Therefore i would propose a solution to make 5 tables and get revenue of Rs 400.

T – This solution looks good. Don't you want to consider solutions where you make tables and chairs? Can you get higher revenue through such solutions?

S – I will be happy if the revenue increases

T - You want to increase the revenue. You get Rs 400 by making 5 tables. You consider making 4 tables. In this process you will make some more resources available. Try making as many chairs as you can by using these resources. You may get more revenue.

S – Ok. Let me make 4 tables. This gives me revenue of Rs 320. I use 12 units of material and 8 units of time. I have 3 units of material and 7 units of time. With 3 units of material, I can make 1.5 chairs and with 7 units of time I can make $7/3 = 2.33$ chairs. I can make 1.5 chairs. I have to reduce it to 1 chair since I cannot make fractional number of chairs. I get additional revenue of Rs 70 which gives me Rs 390. The revenue does not increase.

T – You were correct when you finalized that you could make 1.5 chairs with the available resources after making 4 tables. Later you reduced it to one chair and hence your additional revenue was 70. If you had chosen the solution to make 1.5 chairs, your additional revenue would have been 105 and the total revenue would be 425, which is higher.

S – But commonsense tells me that I cannot make 1.5 chairs. The number of tables and chairs that i make should be integers and cannot be fractions.

T - I agree with you. At present I am going to ask you to assume that you can make fractional number of tables and chairs. We will discuss the correctness of this assumption later.

S – I am reluctant to make this assumption because it is incorrect. I accept your view and let us proceed with a solution of 4 tables and 1.5 chairs with revenue of Rs 425.

T – Now that the revenue has increased, do you want to reduce the number of tables further?

S – If I make three tables, I have 6 units of material and 9 units of time available to make chairs. I can make 3 chairs. Now my revenue is Rs 450. I also have a solution without fractions. I think this should be the best solution.

T – You should observe that if you had concluded earlier that the revenue was not increasing because you could make only 1 chair when you made 4 tables, you would not have reached this solution.

S – I agree with you. I could get to this solution only if i make an assumption that fractional numbers are possible. Else i would not have arrived at this solution.

T – Can you increase the revenue further if you consider making 2 tables?

S – I can try that option. If i make only 2 tables, i have 9 units of material and 11 units of time. Using these I can make 3.66 chairs. My revenue is $160 + 256.66 = 416.66$. The revenue reduces. Should I now consider making 1 table?

T – You need not look at that case. Once it reduces, it cannot increase further. In fact after considering 1 table, you would have considered zero tables. You could make 5 chairs and the revenue would be 350.

S – OK. Now, can I assume that the best possible revenue is 450 when I make 3 tables and 3 chairs?

T – You have got the best solution. Let us see if we use all the resources or if some have not been used.

S – When i make 3 tables and 3 chairs I have used all the 15 units of material and 15 units of time. When i make only 5 tables (and no chairs), I use 15 units of material and 10 units of time. I don't use all the time. When i make 5 chairs and no tables, i use only 10 units of material but 15 units of time. Some material is unused.

T – Yes. Do you want to generalize that you get the highest revenue if you make products and use all the resources fully?

S – Why not? It only looks fairly obvious. In fact, I need not have started with 5 tables and gradually increased the chairs by reducing the number of tables. I could have solved $3X + 5Y = 15$ and $5X + 3Y = 15$ and got the solution with $X = 3$ and $Y = 3$. I have the highest revenue and I use all the resources.

T – You should have done similar problems in high school. That is how you could quickly relate to solving equations.

S – In fact I could have done this by graph also. I remember learning this in high school.

T – I have to say that what you concluded just now may not be correct always. For example, let us assume that the revenue per table is 120 and revenue per chair is 70. What happens to the total revenue?

S – Five tables give revenue of Rs 600; five chairs give revenue of Rs 350 while 3 tables and 3 chairs give revenue of 570.

T – Now you observe that the solution with 3 tables and 3 chairs and revenue of 570 is not the best. The best solution is to make 5 tables, earn revenue of 600 and not use all resources fully.

S – I agree. The extent of resource utilization depends on the solution that we consider and the best solution with the highest revenue also depends on the individual revenues.

T – Your approach of starting with 5 tables and reducing them progressively to get the best solution worked for this problem. Suppose we had a different problem where start with 50 tables or 50 chairs. Would you follow the same approach of reducing the tables by 1 and checking if you get higher revenue?

S – That would make it lengthy and time consuming. Would I still get the best solution with the highest revenue?

T – Yes. You will get the best solution but as you said, it will be lengthy and time consuming. The answer lies in something you said earlier. Can you recollect the graphical way of doing this?

S – Yes. We drew the graphs and we found out the intersection points and computed the revenue.

T – This method is called the graphical method and we will discuss this in detail in this chapter. Suppose you make 3 products instead of two, how will you approach it on a graph?

S – I can draw a graph only for two products. I can think of one product and find the highest revenue if I make one product. If I reduce the quantity, I can consider making either one of the remaining product. This gets complicated and I can end up considering too many solutions before I determine the best solution with the highest revenue.

T – The problem does get difficult and time consuming as the number of products or variables increases.

S - How do we solve the problem in such cases?

T – We can solve it algebraically. We also have a special algorithm called the Simplex algorithm about which we will learn in the chapter.

S – One last question. You made me assume that we could make a fractional number of tables and chairs. How do you justify this?

T – It is practical that the number of tables and chairs actually made have to be integers. We cannot obviously make four and a half tables and more importantly cannot sell them. If I can earn Rs 80 for a table, I cannot earn Rs 40 for half a table in practice. However, the same proportionality assumption makes us assume that we can make and sell fractional quantities. If we define these numbers to integers we have to solve a different class of problems. We will learn about these in this chapter and in this book.

S – Thank you very much. This exercise was indeed interesting. I look forward to some more exercises of this kind.

T – Thank you very much for participating in this exercise. We will meet very soon for the next exercise.

In this chapter we will discuss Linear Programming (LP) formulation and applications. We define a Linear Programming problem, state the assumptions, introduce notations and terminology and provide formulations for several simple and involved applications. We also

learn to solve Linear Programming problems using the graphical method and the Microsoft excel solver. We concentrate more on typical formulations and understanding the solutions than on the details of the algorithms that solve the LP problems. These are explained through numerical illustrations. We also provide some insight into duality and explain concepts of dual through numerical illustrations and solutions using the excel solver.

Linear Programming is perhaps amongst the widely used techniques to solve managerial problems. It started with applications in defense and over the last sixty years has been applied extensively in manufacturing, management, public systems, defense, education, health, infrastructure, logistics and distribution, port handling, agriculture, and communication systems to name a few areas.

We begin our study on Linear Programming formulations through an example.

Illustration 1.1

“Fresh and tasty” bakers make cakes and pastries. On a Sunday morning they have to make ‘plum cake’ and “fruit pastries”. They have 59 units of flour and they use 5 units of flour per cake and 4 units per pastry. They also have 46 units of time in the oven that day and each cake requires 4 units in the oven and each pastry requires 3 units of time in the oven. They sell each cake for Rs 32 and each pastry for Rs 25. How many cakes and pastries they should make to maximize their sale?

SOLUTION:

We first create a mathematical model and then solve the model to find out the number of cakes and pastries the baker should make.

The baker has to decide the number of cakes and pastries to make. Let us define X as the number of cakes made and Y as the number of pastries made.

Since they sell a cake for Rs 32 and a pastry for Rs 25, their revenue (if they can sell all the cakes and pastries made) is Rs $32X + 25Y$. Since they wish to maximize the revenue, they want X and Y to

Maximize $32X + 25Y$

If they can sell all the cakes and pastries that they can make and if there are no conditions or restrictions, they will make infinite amount of cakes and pastries. There are limits on the availability of flour and time that will restrict the amount of cakes and pastries made.

Since each cake requires 5 units of flour and each pastry requires 4 units of flour, the total requirement of flour is $5X + 4Y$. The amount of flour available is 59 and therefore X and Y should be such that $5X + 4Y$ do not exceed 59. This is written as

$$5X + 4Y \leq 59$$

Since each cake requires 4 units of time and each pastry requires 3 units of time, the total requirement of time is $4X + 3Y$. The amount of time available is 46 and therefore X and Y should be such that $4X + 3Y$ do not exceed 46. This is written as

$$4X + 3Y \leq 46$$

(The reader should note that the \leq inequality takes care that the requirement does not exceed the availability. We do not use a strict $<$ there because this would prevent all the available resources from being used).

Finally we do not want to produce a negative number of cakes and pastries. The production quantities have to be non negative. This is written as

$$X, Y \geq 0$$

(The reader should note that it is not absolutely necessary to produce both the products. The only condition is that the production quantities should not be negative and this condition is called the non negativity restriction).

The mathematical model is to determine X and Y to

Maximize $32X + 25Y$

Subject to

$$5X + 4Y \leq 59$$

$$4X + 3Y \leq 46$$

$$X, Y \geq 0.$$

TERMINOLOGY

In the above model we used variable names X and Y to denote the number of cakes and pastries made. These are the variables that represent the decisions in the problem. They are called **decision variables**.

The purpose of the model is to find out the values of the decision variables such that the function $32X + 25Y$ is as large as possible. In other words, the objective is to find out X and Y that maximizes $32X + 25Y$ (or makes it as large as possible). The function $32X + 25Y$ is called the **objective function** because it represents the objective of the model which is to maximize it.

The conditions or limitations that restrict the values that the decision variables can take are called the **constraints**. We have two constraints that are $5X + 4Y \leq 59$ and $4X + 3Y \leq 46$.

There is an additional condition that the decision variables should be non negative. These are called **non negativity restrictions** on the decision variables. These are the conditions $X, Y \geq 0$.

LINEAR PROGRAMMING MODEL

Our modelling process was carried out in the following steps:

1. Identify the decision variables.
2. Write the objective function (as a function of one or more decision variables). The objective either maximizes or minimizes the function.
3. Write the constraints (the left hand side is a function of one or more decision variables and the right hand side is usually a known constant). The sign of the inequality that relates the left hand side (LHS) and the right hand side (RHS) is usually a \leq , \geq or $=$.
4. Write the non negativity restrictions on the decision variables.

We followed steps 1 to 4 in our formulation. We also observe that we have modelled the objective function as a linear function of the decision variables. Both the constraints are linear inequalities. Our model is a **Linear Programming (LP) model** because both the objective function and the constraints are linear functions of the decision variables.

ASSUMPTIONS IN LINEAR PROGRAMMING (LP)

1. Additivity – the total revenue is the sum of the revenues generated by the sale of cakes and pastries.
2. Proportionality (or multiplicity): If each cake fetches revenue of Rs 32, X cakes fetch a revenue of 32X.
3. Divisibility– If each cake fetches revenue of Rs 32, making half a cake is allowed and it fetches revenue of 16.
4. Deterministic – All the values and coefficients are known in advance with certainty.

USUAL NOTATION

Since we had two decision variables, we can use X and Y to represent them. If we have a large bakery that makes 40 products (say), we would be at a loss to define notation for these. It is therefore customary to use X_1 to X_n if we have n products. We therefore use X_1 and X_2 to represent the decision variables in our case. Sometimes X_{ij} and X_{ijk} are used to represent the decision variables depending on the situation.

Our final LP model for the baker's problem is

Maximize $32X_1 + 25X_2$

Subject to

$$5X_1 + 4X_2 \leq 59$$

$$4X_1 + 3X_2 \leq 46$$

$$X_1, X_2 \geq 0.$$

Illustration 1.2

Sweet and softy Bakers make two products biscuits (in packets) and cakes. Their demand for biscuit packets and cakes are 20 and 5 respectively. They have 40 hours of regular time and it takes 2 hours to make a biscuit packet and 3 hours to make a cake. They wish to meet the demand and since the regular time available is not enough to meet the demand, they decide to employ overtime. To meet worker requirements they have to employ exactly 25 hours of overtime. The cost of making a biscuit packet by regular time and overtime it costs Rs 8 and 10 respectively. It also costs Rs 6 and 7 respectively to make a cake by regular time and overtime. How many units of biscuit packets and cakes do they produce by the two modes to minimize total cost of production? They also wish to talk to the workers to provide at least 10 hours of overtime.

We have to decide the quantities of biscuit packets and cakes to be made using regular time and over time respectively. The objective is to minimize the cost of production.

Let X_1 and X_2 represent the number of biscuit (packets) made using RT and OT respectively

Let Y_1 and Y_2 represent the number of cakes made using RT and OT respectively.

Since it costs Rs 8 and 10 to make a biscuit packet by regular time and overtime and it costs Rs 6 and 7 respectively to make a cake by regular time and overtime, the objective function is to

$$\text{Minimize } 8X_1 + 10X_2 + 6Y_1 + 7Y_2$$

The demand for biscuits has to met. Since X_1 packets are made using RT and X_2 by OT, total production is $X_1 + X_2$. This cannot be less than the demand of 20. The constraint is

$$X_1 + X_2 \geq 20$$

Similarly the constraint for cakes is

$$Y_1 + Y_2 \geq 5$$

X_1 biscuits and Y_1 cakes are to be made using RT. It takes 2 hours to make a biscuit packet and takes 3 hours to make a cake by RT. The constraint is

$$2X_1 + 3Y_1 \leq 40$$

It takes 2 hours to make a biscuit packet and takes 3 hours to make a cake by OT. Exactly 25 hours of OT has to be used. The constraint is

$$2X_2 + 3Y_2 = 25$$

The non-negativity restrictions are given by

$$X_1, X_2, Y_1, Y_2 \geq 0$$

The final formulation is

$$\text{Minimize } 8X_1 + 10X_2 + 6Y_1 + 7Y_2$$

Subject to

$$X_1 + X_2 \geq 20$$

$$Y_1 + Y_2 \geq 5$$

$$2X_1 + 3Y_1 \leq 40$$

$$2X_2 + 3Y_2 = 25$$

$$X_1, X_2, Y_1, Y_2 \geq 0$$

If the negotiations with the workers take place, the constraint $2X_2 + 3Y_2 = 25$ is replaced by the constraint $2X_2 + 3Y_2 \geq 10$.

In the above formulation we have seen a minimization objective. We have also come across constraints that are equations and/or inequalities of the \geq and \leq type.

In LP problems, the objective is to maximize or minimize a linear function of the decision variables. The constraints are linear and are of the \geq , \leq and $=$ type. The variables are all ≥ 0 .

Sometimes we may formulate variables that cannot be definitely defined as ≥ 0 . For example, if we have a decision variable that represents the profit made and if profit is defined as the difference of two variables (X_1 is the profit while X_2 is the sale and X_3 is the cost) $X_2 - X_3$, we may have positive or negative or zero profit. In such cases we define the variable as unrestricted. We consider the game theory example which has an unrestricted variable.

Illustration 1.3

Consider two manufacturers (A and B) who are competitors for the same market segment for the same product. Each wants to maximize their profit market share and adopts two strategies. The gain (or pay off) for A when A adopts strategy i and B adopts strategy j is given by a_{ij} . A 2 x 2 matrix of gains for A shown in [Table 1.1](#)

[Table 1.1](#) – Payoff matrix

1	-3
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-1	2
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During a given time period T, both A and B have to mix their strategies. If A plays only strategy 1, then B would play strategy 1 to gain, which A would not want. Each therefore wants to mix their strategies so that they gain maximum (or the other loses maximum).

Let us consider A's problem of trying to maximize his return. Let us assume that A plays strategy 1 x_1 proportion of times and plays strategy 2 x_2 proportion of times. We have

$$x_1 + x_2 = 1$$

If B consistently plays strategy 1, then A's expected gain is $x_1 - x_2$. If B consistently plays strategy 2, A's expected gain is $-3x_1 + 2x_2$.

Player A also knows that B will not consistently play a single strategy but would play his (her) strategies in such a way that A gains minimum. So A would like to play his (her) strategies in proportions x_1 and x_2 to maximize this minimum return that B would allow.

A's strategy is called **Maximin** strategy and would maximize the minimum of $x_1 - x_2$ and $-3x_1 + 2x_2$.

Let us define u as the minimum of $x_1 - x_2$ and $-3x_1 + 2x_2$

A's problem is to

Maximize u

Subject to

$$u \leq x_1 - x_2$$

$$u \leq -3x_1 + 2x_2$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0, u \text{ unrestricted in sign.}$$

(This is because A can end up making an expected loss. Therefore u can be positive, zero or negative in sign).

Can you formulate a problem for minimizing the maximum loss for B if B plays the two strategies y_1 and y_2 proportion of times?

SOLUTION – GRAPHICAL METHOD

Let us try to solve the baker's LP problem given below:

Maximize $32X_1 + 25X_2$

Subject to

$$5X_1 + 4X_2 \leq 59$$

$$4X_1 + 3X_2 \leq 46$$

$$X_1, X_2 \geq 0.$$

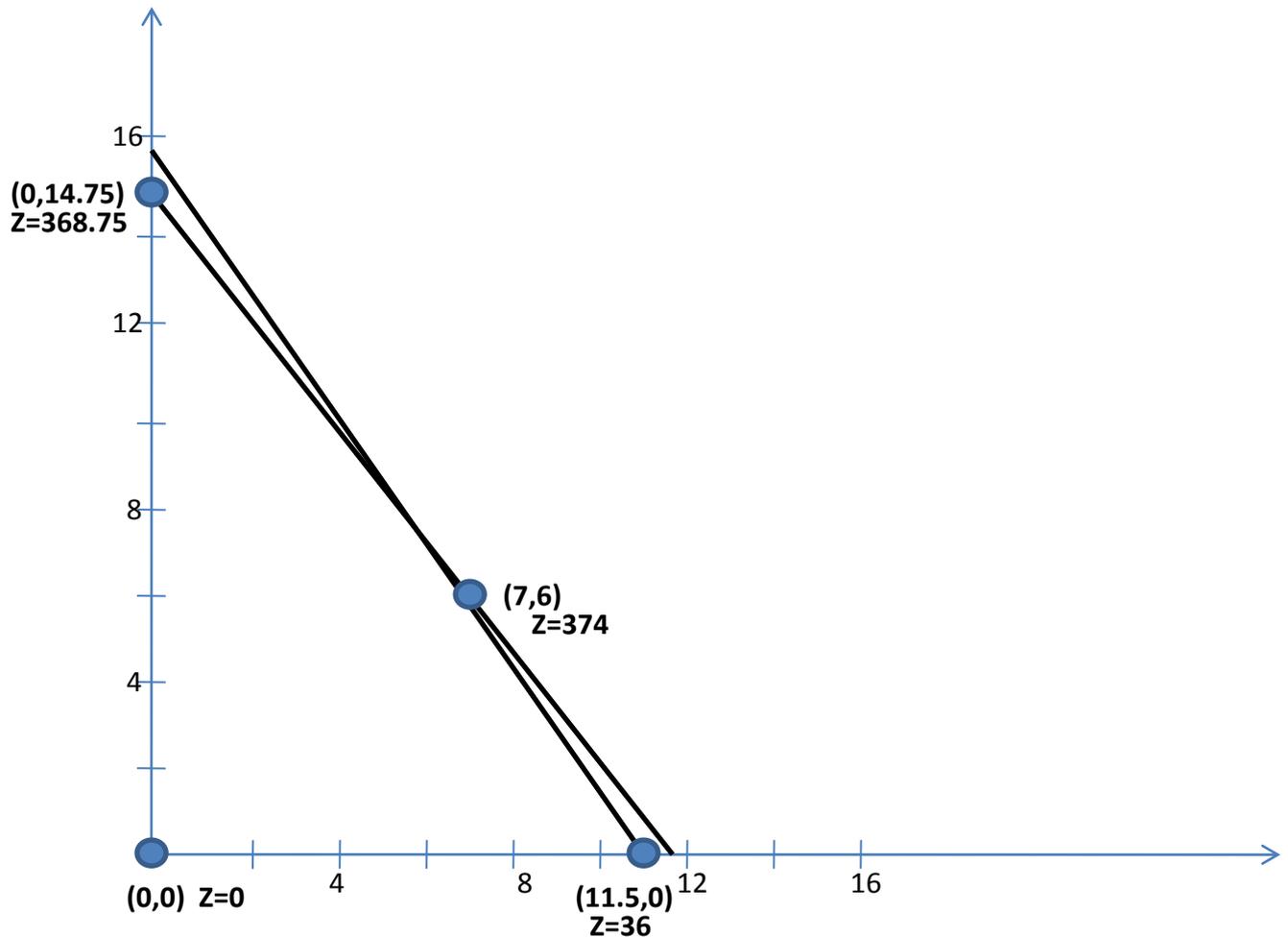


Figure 1.1 – Graphical solution for Illustration 1.1

Since there are only two variables, we can use the graphical approach to solve the problem. We plot the two inequalities on a graph sheet. Figure 1.1 shows the graphs for $5X_1 + 4X_2 \leq 59$ and for $4X_1 + 3X_2 \leq 46$. Since we have used only the first quadrant we ensure that $X_1, X_2 \geq 0$.

The shaded region shows the set of points that satisfy both the inequalities. The shaded portion represents the **feasible region** where every point in the feasible region satisfies all the constraints including the non-negativity restrictions). A point (or solution) is a **feasible solution** if it satisfies all the constraints. Examples of feasible solutions are $(0, 0)$, $(8, 4)$, $(11, 0)$, $(0, 14.75)$ etc.

(An important issue arises here. We have defined X_2 as the number of pastries to be made. Can X_2 take a value 14.75? Can we make a fractional number of pastries? Should X_2 be an integer?

At present we have assumed X_1 and X_2 to be continuous variables and we have not placed an explicit integer restriction on X_1 and X_2 . We therefore do not restrict them to be integers. Restricting them to be integers leads us to a different class of problems called Integer programming problems and we will learn about them later.

As far as LP is concerned, we assume that all variables are continuous and can take fractional values. We worry about the implementation aspects after we solve the LP).

Since every feasible solution satisfies all the constraints (including the non negativity) we can implement all of the solutions (From now on, if we say that a solution satisfies all the constraints, we include the non negativity restrictions as well). We obviously want the final solution to be implementable and therefore it has to be feasible. It is the feasible solution that has the highest value of $32X_1 + 25X_2$ (the objective function).

Let us evaluate the objective function value for the feasible solutions. Let Z denote the value of the objective function for the feasible solutions. [Table 1.2](#) shows the values.

[Table 1.2](#) – Feasible solutions

No.	Solution	Objective function
1	(0, 0)	0
2	(8, 4)	356
3	(11, 0)	352
4	(0, 14.75)	368.75

Among the four solutions we would choose (0, 14.75) because it has the highest value of the objective function. We also wish to find out if we can have a solution with higher value of the objective function.

Let us consider the solution (0, 15). This has an objective function value of 375 but violates the constraint $5X_1 + 4X_2 \leq 59$. We do not consider this solution because it is infeasible and violates a constraint.

Let us consider the solution (16, -6). This satisfies both the constraints $5X_1 + 4X_2 \leq 59$ and $4X_1 + 3X_2 \leq 46$. It has an objective function value of 362. It is not considered because it violates the non negativity restriction.

We therefore know that the best solution is a feasible solution with the best possible (highest) value of the objective function.

Let us consider the solution (8, 4). This solution is feasible because it is inside the feasible region. The objective function value is 356. We can fix $X_1 = 8$ and increase X_2 by moving to the right of this point till we reach the point (8, 4.666) to get an objective function value of 372.67. We can fix $X_2 = 4$ and increase X_1 by moving above this point till we reach the point (8.5, 4) to get an objective function value of 372.

We could generalize and say that for every point inside the feasible region, it is possible to move to the right (left) or move above (below) depending on the nature of the objective function (maximize or minimize) or the sign of the coefficients (positive or negative) and obtain a point on the boundary of the feasible region with a better value of the objective function.

We can also generalize that the best solution is a point that lies on the boundary of the feasible region.

Let us consider the point (8, 4.666) with an objective function value of 372.67. This point lies on the line $4X_1 + 3X_2 = 46$. The slope of this line is 0.75. The slope of the objective function line is $25/32 = 0.78$. Since the slope of the objective line is higher we can increase X_2 to get points that satisfy $4X_1 + 3X_2 = 46$ and $5X_1 + 4X_2 \leq 59$. This can increase X_2 to 6 and bring down X_1 to 7 to get the point (7,6) with an objective function value of 374. Here $5X_1 + 3X_2$ becomes equal to 59. Increasing X_2 further and retaining $4X_1 + 3X_2 = 46$ would violate $5X_1 + 4X_2 \leq 59$. The best solution is (7, 6) with **Z = 374**.

The best solution is called the **optimal solution or optimum solution** and is a corner point solution in the feasible region.

Therefore it is enough if we determine the corner points and compute the objective function at these points. [Table 1.3](#) shows these computations.

Table 1.3 – Corner point solutions

No.	Solution	Objective function
1	(0, 0)	0
2	(0, 14.75)	368.75
3	(11.5, 0)	368
4	(7,6)	374

The best among the corner point solutions is (7,6) with $Z = 374$. This is the optimum solution.

Another way to get to the optimal solution is to draw iso revenue lines. The objective function is $32X_1 + 25X_2$. Draw a line $32X_1 + 25X_2 = 200$ and $32X_1 + 25X_2 = 300$. The second line shows that every point in the feasible region as well as in the line $32X_1 + 25X_2 = 300$ has an objective function value of 300. Since we want to get a feasible point that maximizes $32X_1 + 25X_2$, we draw revenue lines by increasing the value (draw lines parallel to $32X_1 + 25X_2 = 300$) till it just touches the last point in the feasible region. This will be a corner point (if the slope of the objective function is different from the slopes of the constraints). In our example it is the corner point (7, 6) with $Z^* = 374$. (We use Z^* to denote the objective function value at the optimum).

STEPS IN THE GRAPHICAL METHOD

1. Draw the constraints and obtain the feasible region.
2. Identify the corner points of the feasible region.
3. Evaluate the objective function at the corner points and identify the optimum solution.

The reader should note that the graphical method works extremely well when we have two decision variables. When the number of decision variables increases, it becomes difficult to use the graphical method to get the optimum solution. We use the Simplex algorithm to solve LP problems with more than 2 variables. Several solvers are available and the most popular among them is the solver available as an add on to Microsoft excel.

SOLVING LP PROBLEMS USING EXCEL SOLVER

In this chapter we explain the use of the excel solver to solve LP problems. We consider the same example that we used to illustrate the graphical method. The example is:

Maximize $32X_1 + 25X_2$

Subject to

$$5X_1 + 4X_2 \leq 59$$

$$4X_1 + 3X_2 \leq 46$$

$$X_1, X_2 \geq 0.$$

INSTALLING THE SOLVER INTO MICROSOFT EXCEL

The following steps help us in installing the solver into Microsoft excel screen:

1. Go to the Windows icon, the top leftmost icon in MS Excel
2. Click on excel options
3. Go to add in (on the left side of the new screen)
4. Choose solver add in and click on the GO button
5. It will install the solver.
6. If you click on Data from the main excel page, you can see the solver icon on the right hand side.
7. Now you are ready to solve LPs using this solver.

SOLVING THE GIVEN LP

Let us create an excel file with any name (say, example1.xlsx). [Figure 1.4](#) shows the initial entries

[Table 1.4 – Initial entries in input](#)

	x1	x2	
	0	0	
obj fn	0		
	0	<=	59

	0	<=	46
	0	>=	0
	0	>=	0

1. Enter the two variable names (say, X1 and X2) in positions B2 and B3. Let the initial values that we give to these be 0 and 0 (shown under B4 and C4).
2. We show the objective function value under any position (say, B6). In position B6 we write $=32*B4+25*C4$ so that the value of the objective function is stored there. Now since B4 and C4 are zero, B6 value will be computed as 0. (We could type the values 32 and 25 in two positions, say C6 and D6 and write B6 as $=C6*B4+D6*C4$)
3. We have two constraints and two non negativity restrictions.
4. We write the values 5 and 4 in two positions, say E2 and F2 and write the left hand side of the first constraint in B8 as $=E2*B4+F2*C4$. The value at B8 is zero since B4 and C4 are zero. We write the RHS value of 59 in D8.
5. We write the values 4 and 3 in two positions, say E3 and F3 and write the left hand side of the first constraint in B9 as $=E3*B4+F3*C4$. The value at B9 is zero since B4 and C4 are zero. We write the RHS value of 46 in D9.
6. We have two decision variables that have to take non negative values. We enter $=B4$ in B10 and enter $=C4$ in B11. These represent the LHS values of the non negativity restriction. The RHS values of zero are entered in D10 and D11.
7. We are ready to enter the problem into the solver from the excel sheet.

SOLVER ENTRIES AND SOLUTIONS

1. Click on the SOLVER. A new window titled Solver Parameters will appear.
2. Click on the space for setting target cell and click B6. We have defined our objective function in B6.
3. Click on the button max because we want to maximize the objective function value (Else click on min if we are solving a minimization objective).
4. There is a message “by changing cells” and space to define these. This is the place where the decision variables have to be defined. Since the values of the decision variables are in B4 and C4, click on B4, use the shift and take it to C4. Now both B4 and C4 are marked. It will show $\$B\$4:\$C\4 indicating the values of B4 and C4 are the decision variables.
5. Now we have to enter the constraints. There is space under “subject to the constraints”. Click on the add button to add constraints. Another window appears.
6. Under cell reference click on B8. It appears as $\$B\8 which represents the LHS of the first constraint. Click on the \leq and click on D8 under the constraint. It appears as $\$D\8 . Click on add and the first constraint is entered.
7. Add the second constraint by making B9 \leq D9.
8. The two non-negativity constraints are added as B10 \geq D10 and B11 \geq D11.
9. Now enter “Solve”. We get the solution shown in [Table 1.5](#)

Table 1.5 – Final solution

	x1	x2	
	7	6	
obj fn	374		
	59	<=	59
	46	<=	46
	7	>=	0
	6	>=	0

The solution is $X_1 = 7$ and $X_2 = 6$ (shown in positions B4 and C4) with objective function value = 374 (shown in position B6). For this solution the values of the LHS of the constraints are shown in B8 and B9 and the values of the LHS for the non negativity (the values of the variables themselves) are shown in B10 and B11.

COMMON QUESTIONS AND DOUBTS

1. Is it necessary that I have to produce both the products? Can I have a solution where I produce only one of them or a subset of the defined products?

It is not necessary that if there are two decision variables (two products), we need to produce both of them. It depends on the constraints and the objective function. Consider the same problem and change the objective function to $40X_1 + 25X_2$. We change the value of B8 as $=40*B4+25C4$ and solve the problem again. The optimum solution is $X_1 = 11.5$ with objective function value of 460. We make only cakes and we do not make pastries.

2. There were two constraints in the original problem. We produced two products? Is the number of products made same as the number of constraints?

It is not necessary that we produced two products because we had two constraints. The answer to the question is the same as the answer for the previous question. If the objective function becomes $40X_1 + 25X_2$, the solution changes to $X_1 = 11.5$ with $Z = 460$.

3. If there are two constraints and two decision variables, should all the resources be utilized fully?

In our example, each constraint represents a resource. When the objective function is to maximize $32X_1 + 25X_2$, the optimum solution is $X_1 = 7$ and $X_2 = 6$. The resources required for this solution are $5X_1 + 4X_2 = 59$ and $4X_1 + 3X_2 = 46$ units. The resources available are 59 and 46 respectively. This shows that the resources are utilized fully.

If we have 2 decision variables and 2 constraints (resources) and both the variables are in the optimum solution, the resources are utilized fully.

If the objective function is changed to maximize $40X_1 + 25X_2$, the optimum solution is $X_1 = 11.5$ and $X_2 = 0$. The resources required for this solution are $5X_1 + 4X_2 = 57.5$ and $4X_1 + 3X_2 = 46$ units. The resources available are 59 and 46 respectively. This shows that the first resource is not utilized fully while the second resource is utilized fully.

If there are m constraints and fewer than m decision variables are in the solution (with positive values) some resources are not fully utilized. The unutilized resource (with a positive value) is represented by a slack variable in the solution.

When the objective function is changed to maximize $40X_1 + 25X_2$, the optimum solution is $X_1 = 11.5$. Resource 1 is unutilized to the extent of 1.5 units. There are two constraints and there will be two variables in the solution. These are the decision variable $X_1 = 11.5$ and unutilized resource $X_3 = 1.5$.

4. How can the optimum solution leave a resource unutilized? Is it not possible to increase the revenue further by utilizing the rest of the resources?

In the situation where the optimum solution is $X_1 = 11.5$, we have 1.5 units of the first resource unutilized. We cannot increase X_1 because resource 2 is fully utilized. We can increase X_2 and decrease X_1 . This decreases the revenue because the revenue of the first product is higher than that of the second product.

5. If we have two decision variables, a two constraint problem can have both these in the optimum solution. What happens if we have three constraints and two decision variables?

The answer to this question is given in the example 2 where we revisit the bakers problem?

Illustration 1.4 – The Baker Problem Revisited

The baker realizes that he requires space to store the cakes and pastries. The storage requirement for each cake is 3 units and the space for each pastry is 2 units. The total shelf space available is 40 units. What happens to the optimum solution if the objective function is to Maximize $32X_1 + 25X_2$?

We now have to consider an additional constraint that limits the storage space. This constraint is $3X_1 + 2X_2 \leq 40$

The easiest thing to do is to solve the problem again.

To do this we create another constraint. We choose position B12 to indicate the left hand side and write $=3*B4+2*C4$ there and write the RHS value of 40 in D12. We go to the solver and

add the constraint. This will be shown as $B \leq D$. We solve again to get the solution $X_1 = 7, X_2 = 6$ with $Z = 374$.

Before the constraint was introduced, the optimum solution was $X_1 = 7, X_2 = 6$. We substitute the solution to the new constraint and observe that the value of $3X_1 + 2X_2$ for the solution is 33 which is the space required for the optimum solution. This is less than the available space of 40 and hence the present optimum solution will remain optimum.

When a constraint is added to an existing LP, we verify if the present optimum solution satisfies it. If it satisfies the constraint, the optimum solution remains the same after the addition of the constraint.

WHAT HAPPENS WHEN THE CONSTRAINT IS NOT SATISFIED?

We assume that only 30 units of space are available for the cakes and pastries because 10 units are to be reserved to store a large order. Now the constraint becomes $3X_1 + 2X_2 \leq 30$. We modify the RHS of the third constraint (We enter 30 in D12) and solve again. The optimum solution is $X_1 = 1, X_2 = 13.5$ with $Z = 369.5$.

We also note that the introduction of a binding constraint reduces the value of the objective function. *In general, addition of a constraint can only worsen the solution.*

There are only two products and both are produced. It is observed that the second resource is not fully utilized and 1.5 units are left behind. The other two resources are fully utilized. If there are 2 decision variables and three resource constraints, and both products are made, two resources will be fully utilized.

(There is another method called sensitivity analysis where we can start from the previous optimum solution of $X_1 = 7, X_2 = 6$ and proceed to get the new optimum solution after the addition of the new constraint. This method is often used when LP problems are solved by hand using the Simplex algorithm. Since we are using a solver, we can solve the problem again though using sensitivity analysis can be faster computationally. We see some aspects of sensitivity analysis in a later chapter)

CAN I SOLVE LPS BY SUCCESSIVELY ADDING CONSTRAINTS?

The answer is YES. Let us consider the LP

Illustration 1.5

Solve the LP **Maximize $32X_1 + 25X_2$**

Subject to

$$5X_1 + 4X_2 \leq 59$$

$$4X_1 + 3X_2 \leq 46$$

$$X_1, X_2 \geq 0.$$

We consider the objective function and the first constraint only. This means that we have only the first constraint and the non negativity restrictions on two decision variables. In the solver we include

$$\begin{aligned} & \$B\$8 \leq \$D\$8 \\ & \$B\$10 \geq \$D\$10 \\ & \$B\$11 \geq \$D\$11 \end{aligned}$$

We solve the LP to get the optimum solution $X_1 = 11.8$, $X_2 = 0$ with objective function $Z = 377.6$.

We now check if the solution $X_1 = 11.8$ satisfies the second constraint $4X_1 + 3X_2 \leq 46$. The LHS value for the constraint at $X_1 = 11.8$ is 47.2 which exceeds 46. The constraint is violated. We have to add the constraint $4X_1 + 3X_2 \leq 46$ and include

$\$B\$9 \leq \$D\9 in the set of constraints in the solver. We solve to get the optimum solution $X_1 = 7$, $X_2 = 6$ with $Z = 374$.

(If the solution $X_1 = 11.8$ had satisfied $4X_1 + 3X_2 \leq 46$, it would have been optimum to the two constrained LP problem. We had to include the second constraint into the problem and solve again. We ended up solving the problem twice. Since we knew both two constraints at the formulation stage, it is advisable to solve the entire problem in one effort rather than try to solve it progressively by adding constraints).

When one or more constraints are removed from the LP, the resultant problem is a relaxed problem. If the optimum solution to the relaxed LP satisfies the relaxed constraints, it is optimum to the original LP.

6. If we have more decision variables than constraints, can we produce all the products?

Illustration 1.6

Consider the baker's problem. Assume that there is a third product "biscuits" made in packets. Each biscuit packet fetches revenue of Rs 18. To make each biscuit packet, we require 3 units of flour and 2 units of time. Sixty units of flour are available and 44 hours are available. Formulate and solve the problem again?

We have already defined X_1 and X_2 as the amount of cakes and pastries made. X_3 is defined as the number of packets of biscuits made. The LP problem is to

$$\begin{aligned} & \text{Maximize } 32X_1 + 25X_2 + 18X_3 \\ & \text{Subject to} \end{aligned}$$

$$5X_1 + 4X_2 + 3X_3 \leq 60$$

$$4X_1 + 3X_2 + 2X_3 \leq 44$$

$$X_1, X_2, X_3 \geq 0.$$

We modify the excel sheet where we have the baker's problem. A third variable X_3 is introduced in position D4 with value zero. We modify the objective function position B6 to $32*B4 + 25*C4 + 18*D4$.

The LHS of two constraints in B8 and B9 are redefined as $=5*B4+4*C4+3*D4$ and $=4*B4+3*C4+2*D4$ respectively.

The solver icon is clicked. The target cells are defined from B4 to D4. The RHS values are given as 60 and 44. The problem is solved to maximize B6. The solution given in Table 3.3 appears.

	x1	x2	x3
	0	12	4
obj fn	372		
	60	<=	60
	44	<=	44
	0	>=	0
	12	>=	0

The optimum solution is $X_2 = 12$, $X_3 = 4$ with $Z = 372$. We observe that at the optimum we make products 2 and 3 which are pastries and biscuits.

There are three products (decision variables) and two constraints. When we have two constraints we can have a maximum of two products in the solution. *We will not produce all three products in this scenario.*

[Illustration 1.7](#)

Consider the baker's problem with an additional space restriction. The biscuit packet requires 2 units of storage per unit. The additional constraint becomes $3X_1 + 2X_2 + 2X_3 \leq 32$. The resources available are 61, 45 and 32 respectively.

The problem is now to

$$\text{Maximize } 32X_1 + 25X_2 + 18X_3$$

Subject to

$$5X_1 + 4X_2 + 3X_3 \leq 61$$

$$4X_1 + 3X_2 + 2X_3 \leq 45$$

$$3X_1 + 2X_2 + 2X_3 \leq 32$$

$$X_1, X_2, X_3 \geq 0.$$

We add the third constraint into the solver and obtain the optimum solution. The solution is $X_1 = 6$, $X_2 = 1$ and $X_3 = 9$ with $Z = 379$.

Here we have a situation where by adding a constraint we are able to produce all the three products.

(Ordinarily addition of a constraint will only worsen (decrease) the objective function value. We have changed the resource availability and the increase has resulted in an increase in the objective function in spite of an additional constraint).

7. How do I solve minimization problems? Is it any different from solving a maximization problem?

Illustration 1.8

Solve the minimization problem Minimize $8X_1 + 10X_2 + 6Y_1 + 7Y_2$

Subject to

$$X_1 + X_2 \geq 20$$

$$Y_1 + Y_2 \geq 5$$

$$2X_1 + 3Y_1 \leq 40$$

$$2X_2 + 3Y_2 = 25$$

$$X_1, X_2, Y_1, Y_2 \geq 0$$

We set up the problem in excel to solve it. [Table 1.7](#) shows the initialization in the spreadsheet

[Table 1.7](#) – Initial table

	x1	x2	y1	y2
	0	0	0	0
obj func	0			
	0	>=	20	
	0	>=	5	
	0	<=	40	
	0	=	25	
	0	>=	0	
	0	>=	0	
	0	>=	0	

	0	>=	0	
--	---	----	---	--

If the rows are A, B, C etc and the columns are 1, 2, 3 etc, the decision variables are represented in B2, C2, D2 and E2 respectively. They are initialized with value 0.

The position B6 has =B2+C2, position B7 has =C2+D2 and so on. Positions B6 to B13 has the LHS values of the 4 constraints and the four non negativity restrictions. Click on the solver and in the usual way enter the constraints. These are

\$B\$6>=\$D\$6; \$B\$7>=\$D\$7; \$B\$8<=\$D\$8; \$B\$9=\$D\$9; \$B\$10>=\$D\$10; \$B\$11>=\$D\$11; \$B\$12>=\$D\$12; \$B\$13>=\$D\$13.

The third constraint is a <= type constraint since it represents the RT capacity available. The fourth constraint is an equation since the company wants to use exactly 25 units of OT. We solve the problem to get the optimum solution

$X_1 = 15$, $X_2 = 5$ and $Y_2 = 5$. [Table 1.8](#) shows the final solution. The value of the objective function is 205.

[Table 1.8](#) – Final solution

	x1	x2	y1	y2
	15	5	0	5
obj func	205			
	20	>=	20	
	5	>=	5	
	30	<=	40	
	25	=	25	
	15	>=	0	
	5	>=	0	
	0	>=	0	
	5	>=	0	

This solution has produced 5 units of each product using OT since 25 units of OT has to be used. 10 units of RT capacity are unutilized because of this.

The other option that the company has is to convince the employees to accept at least 10 units of OT. This leads us to the constraint $2X_2 + 3Y_2 \geq 10$ instead of $2X_2 + 3Y_2 = 25$. We click on the solver and use the change option to change the sign of the constraint and it becomes \$B\$9>=\$D\$9. The RHS value is changed to 10.

The optimum solution is $X_1 = 20$ and $Y_2 = 5$ with $Z = 195$. Fifteen units of OT is used and RT is used fully. The cost reduces to 195.

8. Should the RHS values of the constraints be positive always?

The RHS values of all the constraints in LP problems have to be non negative. They can be zero for one or more constraints. If the RHS value is negative, we multiply the sign by -1 to make the RHS value positive. The sign can change if the constraint is an inequality.

9. Do all LPs have optimum solutions? Are there special cases?

Illustration 1.9

Solve the LP

$$\text{Maximize } 7X_1 + 5X_2$$

Subject to

$$X_1 - 2X_2 \leq 12$$

$$2X_1 \leq 8$$

$$X_1, X_2 \geq 0$$

We solve this LP using the solver as we did earlier. The solver output is given in [Table 1.9](#). In addition there is a message **“The cell values do not converge”**.

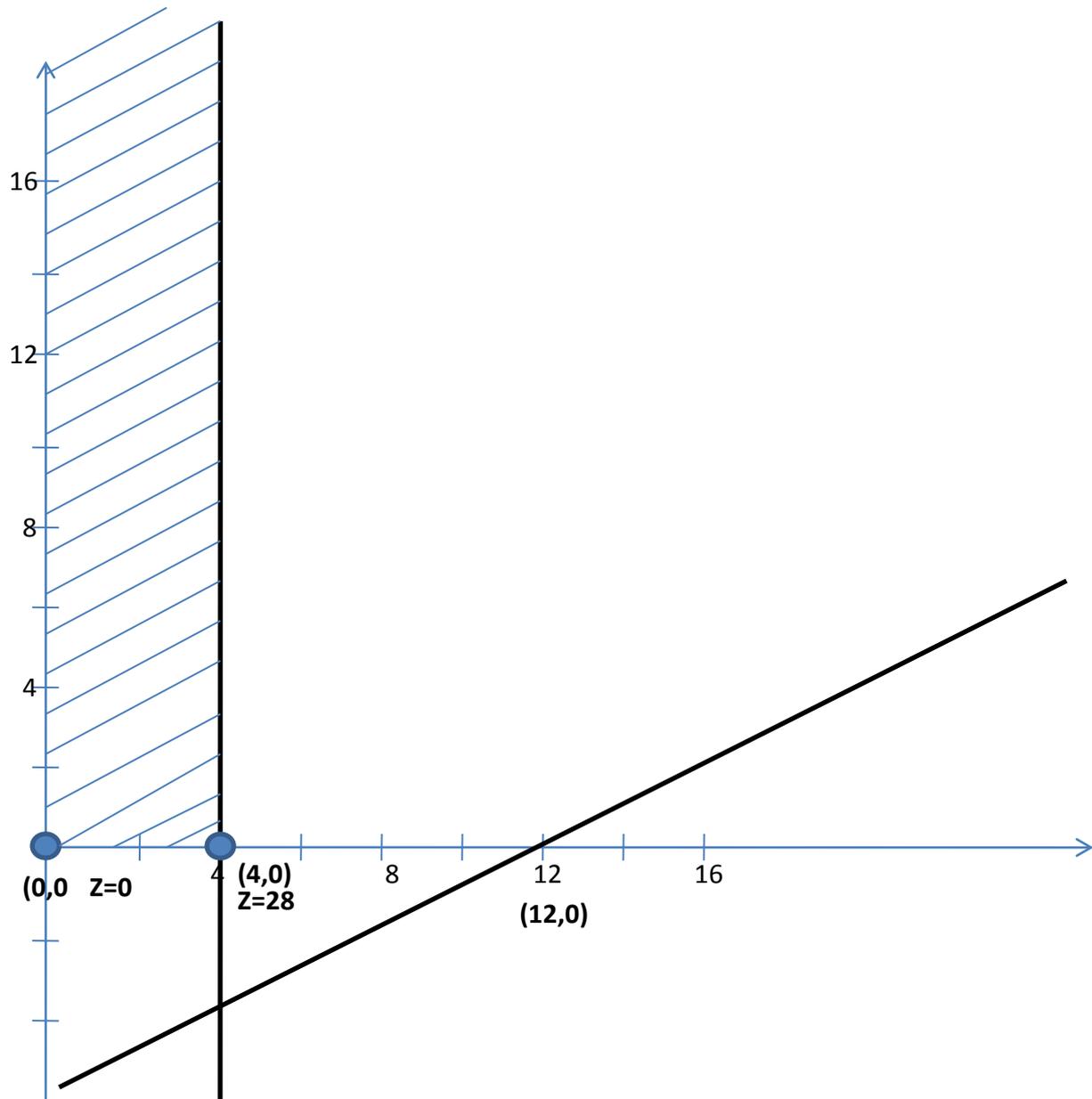
Table 1.9 – Solver output

	x1	x2	
	4	8.24E+15	
	4.12E+16		
	-1.6E+16	<=	12
	8	<=	8
	4	>=	0
	8.24E+15	>=	0

It is observed that variable X_2 has taken a very large value when the computations stopped. When the solver terminates with the message **“The cell values do not converge”**, it means that the LP problem is **unbounded**. This means that the values of one of the variables can be increased to infinity.

The graph for this problem is shown in Figure 1.2. From the graph we can observe clearly that the variable X_2 can be increased to infinity.

LPs formulated from practical situations should not be unbounded. Usually when an LP problem terminates with an unbounded solution, we should check the sign of the coefficients in the constraints. If the problem is formulated out of a practical situation, it is likely that during the formulation the sign of one of the coefficients has been typed wrongly.



If we correct the first constraint to $X_1 + 2X_2 \leq 12$ and solve the problem again, we get the optimum solution $X_1 = X_2 = 4$ and $Z = 48$.

Illustration 1.10

Solve the LP - Maximize $7X_1 + 5X_2$

Subject to

$$2X_1 + 2X_2 \leq 8$$

$$X_1 + X_2 \geq 5$$

$$X_1, X_2 \geq 0$$

When we solve the above LP using the solver, it terminates with the solution shown in [Table 1.10](#). There is also a message “**Solver could not find a feasible solution**”.

[Table 1.10](#) – Solver output

	x1	x2	
	4	0	
	28		
	8	<=	8
	4	>=	5
	4	>=	0
	0	>=	0

We observe that the second constraint is violated when the solver terminated the solution of the problem. This situation is called **infeasibility**. The given LP is infeasible.

The graph corresponding to the constraints is shown in [Figure 1.3](#). From the graph it is observed that there is no feasible region.

LPs formulated from existing situations should not be infeasible. Usually when an LP problem terminates with infeasibility, we should check the sign of the inequalities in the constraints. If the problem is formulated out of a practical situation, it is likely that during the formulation the sign of one of the inequalities has been typed wrongly.

Changing the second constraint to $X_1 + X_2 \leq 5$ we get the optimal solution $X_1 = 4$ with $Z = 28$.

If we observe that the sign of the inequalities are correct and there is no mistake during the entry, it is likely that some values in the RHS have been overstated or understated. If we change the RHS of the first constraint to 18, the optimum solution is $X_1 = 9$ with $Z = 63$.

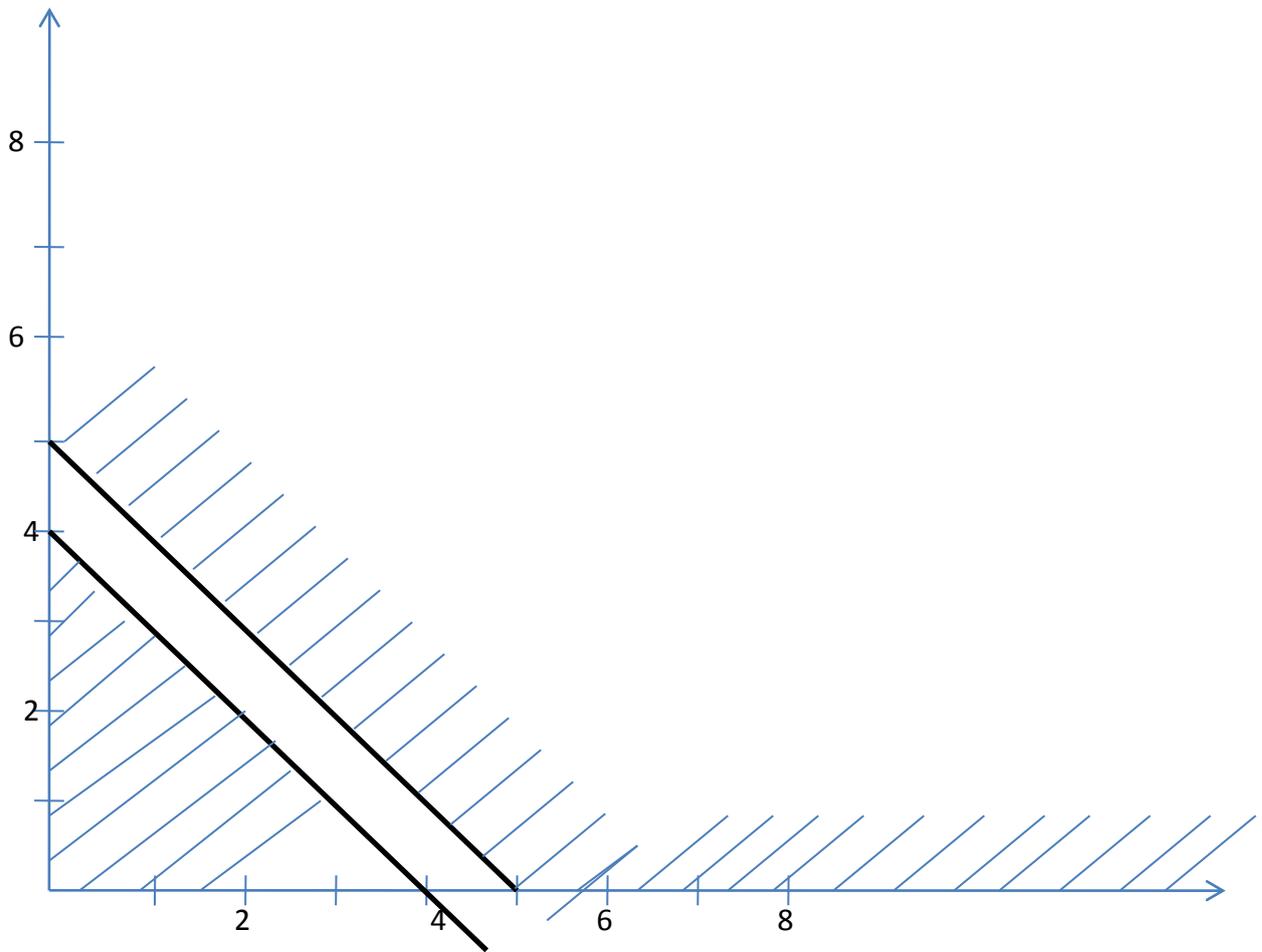


Figure 1.3 – Graph for the constraints in Illustration 1.10

If we are convinced that there is no change in the sign of the inequalities and in the RHS values, it finally means that we are trying to formulate a situation that cannot be implemented or achieved.

Based on the above examples we generalize that

Every LP problem results either in an optimum solution or is unbounded or is infeasible.