

Selected Exercises 4

Disclaimer: Please note that this set of “Selected Exercises” contains additional problems for practise. However, unlike the assignment sheet, this should *not* be turned in! Also, this is not an exhaustive list of exercises. In fact, it is a subset of the exercises given in the book “Nonlinear Dynamics and Chaos”, First Indian Edition (2007) by Steven H. Strogatz. Interested students are encouraged to solve all the exercise problems from the aforementioned book.

1 Examples and Definitions

- 1) For which real values of a does the equation $\dot{\theta} = \sin(a\theta)$ give a well-defined vector field on the circle ?
- 2) For each of the following vector fields, find and classify all the fixed points, and sketch the phase portrait on the circle.
 - a) $\dot{\theta} = 3 + \cos 2\theta$
 - b) $\dot{\theta} = \sin k\theta$

2 Nonuniform Oscillator

- 1) The time required to pass through a saddle-node bottleneck is approximately

$$T_{\text{bottleneck}} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2}.$$

To evaluate this integral, let $x = \sqrt{r} \tan \theta$, use the identity $1 + \tan^2 \theta = \sec^2 \theta$, and change the limits of integration appropriately. Thereby show that $T_{\text{bottleneck}} = \pi/\sqrt{r}$.

- 2) For the following systems, draw the phase portrait as a function of the control parameter μ . Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ .

a) $\dot{\theta} = \frac{\sin \theta}{\mu + \sin \theta}$

b) $\dot{\theta} = \frac{\sin 2\theta}{1 + \mu \sin \theta}$

3 Linear Systems

- 1) [*Damped harmonic oscillator*] The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.
- Rewrite the equation as a two-dimensional linear system.
 - Classify the fixed point at the origin and sketch the phase portrait, clearly showing all the different cases that can occur depending on the relative sizes of the parameters.
 - How do your results relate to the standard notions of overdamped, critically damped and underdamped vibrations?
- 2) Show that any matrix of the form

$$A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix},$$

with $b \neq 0$, has only a one-dimensional eigenspace corresponding to the eigenvalue λ . Then solve the system $\dot{\mathbf{x}} = A\mathbf{x}$ and sketch the phase portrait.