

Selected Exercises 3

Disclaimer: Please note that this set of “Selected Exercises” contains additional problems for practise. However, unlike the assignment sheet, this should *not* be turned in! Also, this is not an exhaustive list of exercises. In fact, it is a subset of the exercises given in the book “Nonlinear Dynamics and Chaos”, First Indian Edition (2007) by Steven H. Strogatz. Interested students are encouraged to solve all the exercise problems from the aforementioned book.

1 Saddle-Node bifurcation

- 1) Consider the system $\dot{x} = r + \frac{x}{2} - \frac{x}{1+x}$.
 - a) Sketch the qualitatively different vector fields that occur as r is varied.
 - b) Determine the critical value of r at which a saddle-node bifurcation occurs.
- 2) [*Unusual bifurcations*] For deriving the normal form of saddle-node bifurcation, we assume that $a = \partial f / \partial r|_{(x^*, r_c)} \neq 0$. To see what can happen if $\partial f / \partial r = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r .
 - a) $\dot{x} = r^2 - x^2$
 - b) $\dot{x} = r^2 + x^2$

2 Transcritical bifurcation

- 1) Show that the system $\dot{x} = rx - \ln(1 + x)$ undergoes a transcritical bifurcation. Determine the value of r at which the bifurcation occurs and sketch the bifurcation diagram of fixed points x^* vs r .

3 Pitchfork bifurcation

- 1) Consider the following system $\dot{x} = rx - \sinh x$.
 - a) Sketch the qualitatively different vector fields that occur as r is varied.
 - b) Determine the critical value of r at which a pitchfork bifurcation occurs.
 - c) Classify the bifurcation as supercritical or subcritical.
- 2) [*Subcritical pitchfork*] Consider the system $\dot{x} = rx + x^3 - x^5$ which exhibits a subcritical pitchfork bifurcation.
 - a) Find algebraic expressions for all fixed points as r varies.
 - b) Sketch the vector fields as r varies. Be sure to indicate all fixed points and their stability.
 - c) Calculate r_s , the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

4 Miscellaneous

- 1) In each of the following systems, determine the value of r at which a bifurcation occurs, and classify the bifurcation as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. Then, sketch the bifurcation diagram of fixed points x^* vs r .
 - a) $\dot{x} = rx - \frac{x}{1+x^2}$
 - b) $\dot{x} = x + \tanh(rx)$
- 2) [*Potentials*] Let $V(x)$ be the potential, in the sense that $\dot{x} = -dV/dx$. For the following systems, sketch the potential as a function of r . Show all the qualitatively different cases, including bifurcation values of r .
 - a) (*Saddle-node*) $\dot{x} = r - x^2$
 - b) (*Transcritical*) $\dot{x} = rx - x^2$
 - c) (*Subcritical pitchfork*) $\dot{x} = rx + x^3 - x^5$