

Module 3

Modeling and state-space formulation of nonlinear systems

Objectives: To be able to model simple mechanical and electrical systems, write their state-space formulation and find the equilibrium set.

Lesson objectives

This module helps the reader in

- developing mathematical models of physical systems with emphasis on simple mechanical systems
- writing down the state-space representation that is amenable for further analysis.

In this module, models of pendulum-like systems will be derived using the total energy of the system. Examples include the pendulum on a cart, rotary pendulum, spherical pendulum and the inertia wheel pendulum. These systems have been studied extensively and continue to generate interest among Physicists, Mathematicians and control theorists.

Suggested reading

- A mathematical introduction to robotic manipulation by Richard M. Murray, Zexiang Li and S. Shankar Sastry

Lecture-10

Of all the motions that may bring a system of material particles from a certain initial position to a given final position (the total energy remaining constant), the actual motion is that for which the action is a minimum. The virtual motions must, therefore, satisfy the principle of energy. They may, on the other hand, take any arbitrary time. According to this conception, the path of a particle is that along which it will reach its final position in the shortest time, if it move with constant velocity, and if frictional forces be absent. Thus, the path is the line of shortest length, that is, for a free particle a straight line. –J. L. Lagrange (1760).

Euler-Lagrange formulation of mechanical systems

In this module we present the equations of motion of well-known benchmark problems in control systems.

The mechanical systems that we study here consist of rigid/flexible linkages which are interconnected in some way. They are given the structure of a set whose points are in one-to-one correspondence with the set of configurations of the system. We denote the configuration set, also called as configuration space by Q . The number of configuration variables depends upon the degrees-of-freedom of the system, defined as follows.

Definition 10.0.33 *The degree-of-freedom of a mechanical system is the number of independent coordinates required to describe the pose or the configuration of the system.*

In many mechanical systems, the space Q is non-Euclidean, for example, the configuration space of a simple pendulum is the Unit circle, that is $Q = \mathbb{S}^1$).

Definition 10.0.34 For a system of m particles with k constraints of the form $g_j(r_1, \dots, r_m)$, $j = 1, \dots, m$ and g'_j s are smooth, we find a set of $n = 3m - k$ variables $(q_1, \dots, q_n)$ and smooth functions $f_1, \dots, f_m$ such that $r_i = f_i(q_1, \dots, q_n) \iff g_j(r_1, \dots, r_m) = 0, i = 1, \dots, m, j = 1, \dots, k$. The q'_i s are called as generalized coordinates for the system.

The number of generalized coordinates is equal to the number of degrees-of-freedom of the system. When the external forces acting on the system are expressed in terms of components along the generalized coordinates, they are termed generalized forces.

There are many methods of modeling mechanical systems viz. using Newton's laws, Euler-Lagrange and Hamiltonian based equations of motion. Nevertheless, all are equivalent and hence yield the same dynamics. We focus on the Euler-Lagrange equations for which we first define the Lagrangian as $\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q)$, where $T(q, \dot{q})$ is the total stored kinetic energy associated with the masses and moments of inertia and the total potential energy $V(q)$ associated with gravitational and stiffness elements.

Theorem 10.0.35 The equations of motion for a mechanical system with generalized coordinates and velocities $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ and Lagrangian $\mathcal{L}(q, \dot{q})$ are given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i, i = 1, \dots, n$$

where F_i is the external force acting on the i^{th} generalized coordinate.

State-space formulation and equilibrium points

An n^{th} -order nonlinear system with m control inputs $u_1, u_2, \dots, u_m$ can be rewritten as n first-order ODEs as

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^m g_i(x) u_i \\ x(t_0) &= x_0 \end{aligned} \tag{10.1}$$

where, $x = (x_1, x_2, \dots, x_n)$ are termed the states of the system, $f(x) \in \mathbb{R}^n$ is termed the drift vector field and $g_i(x) \in \mathbb{R}^n$ are the control vector fields. The vector fields f, g_i are said to be smooth if its components are smooth functions. The underlying space on which the states evolve is termed the state-space of the system. If the system has no control inputs ($u_i = 0, i = 1, \dots, m$), then it is said to be unforced. An important notion associated with unforced system is the equilibrium point denoted by x_e . If $x_0 = x_e$, then the solution $x(t)$ of (10.1) satisfies $x(t) = x_e$ for all $t \geq t_0$. For autonomous systems (where the vector fields f does not depend explicitly on time t) the equilibrium set $\mathcal{E}$

of (10.1) is given by $\mathcal{E} = \{x \in \mathbb{R}^n : f(x) = 0\}$. If $\mathcal{E}$ has a singleton (single point), then the system (10.1) is said to possess an unique equilibrium point. The notion of isolated equilibrium is brought out through the following definition.

Definition 10.0.36 *An equilibrium point $x_e \in \mathbb{R}^n$ of (10.1) is said to be isolated if there exists an $\epsilon > 0$ such that $B(x_e, \epsilon)$ contains no equilibrium points other than x_e .*

If the set $\mathcal{E}$ consist of multiple points and further if each equilibrium point has no other equilibrium in its vicinity, then the system is said to possess multiple isolated equilibrium points. If the multiple equilibrium points are not isolated, then the system is said to have a continuum of equilibrium points. A test to check for isolated equilibrium point is stated as follows.

Proposition 10.0.37 *If $\frac{\partial f}{\partial x}|_{x=x_e}$ is non-singular, then the equilibrium point x_e of (10.1) is an isolated equilibrium point.*

We next present nonlinear models of physical systems along with their state-space formulation and equilibrium sets.

Lecture-11

The pendulum on a cart system is well-known example in the study of nonlinear control. Various control strategies have been devised to swing-up the pendulum and stabilize it at the upward vertical position (analogous to balancing a stick on your palm).

Pendulum on a cart

We consider a pendulum pivoted on a cart as depicted in figure 11.1. The system consists of a pendulum of length l and of point mass m pivoted on a cart of mass M and making an angle of ϕ with the vertical. The cart is actuated with a force F and its displacement is denoted by x . The reference for the potential energy is the horizontal line passing through the origin of the inertial frame. Here, the configuration space is $Q = \mathbb{R} \times \mathbb{S}^1$ and is parameterized by (x, ϕ) . Note that, in topology, the space $\mathbb{R} \times \mathbb{S}^1$ is a cylinder. The

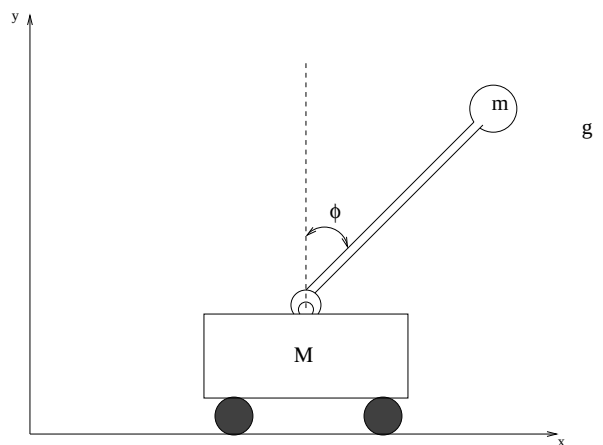


Figure 11.1: Pendulum on a cart system

Lagrangian of the system, is given by

$$\mathcal{L} = \frac{M}{2}\dot{x}^2 + \frac{m}{2}v^\top v - mgl \cos(\phi)$$

where,

$$\begin{aligned} v &= \frac{d}{dt} \begin{bmatrix} x + l \sin(\phi) \\ l \cos(\phi) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \dot{x} + l \dot{\phi} \cos(\phi) \\ -l \dot{\phi} \sin(\phi) \\ 0 \end{bmatrix}. \end{aligned}$$

By collecting the inertial parameters in the following constants $a_1 \triangleq (M + m)$, $a_2 \triangleq ml^2$, $a_3 \triangleq ml$, $a_4 \triangleq mgl$ and defining the control input as $u = F$, the Euler-Lagrange equations are given by

$$\underbrace{\begin{bmatrix} a_1 & a_3 \cos \phi \\ a_3 \cos \phi & a_2 \end{bmatrix}}_M \begin{bmatrix} \ddot{x} \\ \ddot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -a_3 \dot{\phi} \sin \phi \\ 0 & 0 \end{bmatrix}}_C \begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -a_4 \sin \phi \end{bmatrix}}_h = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_G u. \quad (11.1)$$

In robotics, the matrix M in (11.1) is termed the inertia-matrix, the matrix C acting on the vector of generalized velocities $[\dot{x} \ \dot{\phi}]^\top$, in general, consists of the Coriolis and centripetal forces, the vector h represents the gradient of the potential energy and G is the vector of input forces/torques. It is a well-known fact that for rigid-body systems, the following structural properties hold.

- $M = M^\top$ and $M > 0$.
- $\dot{M} - 2C$ is skew-symmetric.

These properties are useful in stability analysis of robotic systems.

With the states defined as $x = [x_1 \ x_2 \ x_3 \ x_4]^\top = [x \ \phi \ \dot{x} \ \dot{\phi}]^\top$ and $u = F$, the state-space model is given by

$$\dot{x} = f(x) + g(x)u \quad (11.2)$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{1}{\Delta}(a_2 a_3 x_4^2 \sin x_2 - a_3 a_3 \sin x_2 \cos x_2) \\ \frac{1}{\Delta}(-a_3 x_4^2 \sin x_2 \cos x_2 + a_1 a_4 \sin x_2) \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ \frac{a_2}{\Delta} \\ \frac{-a_3}{\Delta} \cos x_2 \end{pmatrix},$$

where $\Delta = a_1 a_2 - a_3^2 \cos x_2 > 0$. The equilibrium set is given by $\mathcal{E} = \{x : (x_1, x_2, x_3, x_4) = (*, 0, 0, 0)\} \cup \{x : (x_1, x_2, x_3, x_4) = (*, \pi, 0, 0)\}$. Here $*$ indicates that the cart position is arbitrary.

Points to ponder

- Verify the structural properties of the system?
- Is it possible to compute the equilibrium points of a rigid-body system by examining its potential function? Hint: Look at the gradient of the potential function.

Lecture-12

12.1 Furuta pendulum

We consider the Furuta pendulum, which is an inverted pendulum on a rotary arm as depicted in Figure 12.1, where l is the length of the pendulum, R is the radius of the arm, q_2 is the angle made by the pendulum from the vertical, q_1 is the angle of the mass M from a fixed vertical plane, m is the point mass of the pendulum, g is the acceleration due to gravity and u is the torque applied at the shaft joined to the arm. The total kinetic

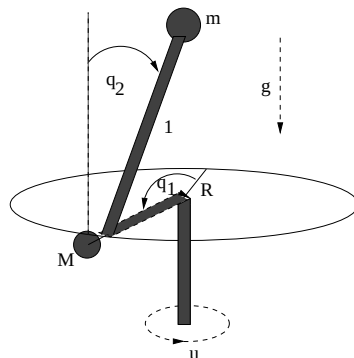


Figure 12.1: Furuta pendulum

energy is given by

$$\begin{aligned} K.E &= K.E_{arm} + K.E_{pendulum} \\ &= \frac{M}{2} R^2 \dot{q}_1^2 + \frac{m}{2} v^\top v \end{aligned}$$

where, v (see figure 12.2) is given by

$$v = \begin{bmatrix} -R\dot{q}_1 \sin q_1 \\ R\dot{q}_1 \cos q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{q}_1 & -\dot{q}_2 \sin q_1 \\ \dot{q}_1 & 0 & \dot{q}_2 \cos q_1 \\ \dot{q}_2 \sin q_1 & -\dot{q}_2 \cos q_1 & 0 \end{bmatrix} \begin{bmatrix} (l \sin q_2) \sin q_1 \\ -(l \sin q_2) \cos q_1 \\ l \cos q_2 \end{bmatrix}.$$

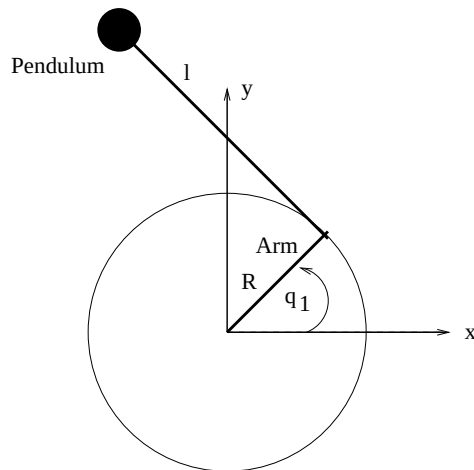


Figure 12.2: Top view of the setup

Collecting the inertial parameters in the constants $a_i, i = 1, \dots, 4$ defined by

$$\begin{aligned} a_1 &= (M + m)R^2 & a_2 &= ml^2 \\ a_3 &= mRl & a_4 &= mgl. \end{aligned}$$

the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(a_1 + a_2 \sin^2 q_2) \dot{q}_1^2 + a_3 \dot{q}_1 \dot{q}_2 \cos q_2 + \frac{a_2}{2} \dot{q}_2^2 - a_4 \cos q_2.$$

The Euler-Lagrange equations are given by

$$\begin{bmatrix} a_1 + a_2 \sin^2 q_2 & a_3 \cos q_2 \\ a_3 \cos q_2 & a_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2a_2 \dot{q}_1 \dot{q}_2 \sin q_2 \cos q_2 - a_3 \dot{q}_2^2 \sin q_2 \\ -a_2 \dot{q}_1^2 \sin q_2 \cos q_2 - a_4 \sin q_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Inertia Wheel Pendulum

The inertia wheel pendulum consists of an unactuated pendulum to which an inertia wheel is attached at the free end (see Figure 12.3). The pendulum angle is θ_1 and the inertia wheel angle is θ_2 (both angles are measured clockwise). With the states defined as $x = [x_1 \ x_2 \ x_3 \ x_4]^\top = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^\top$ and $u = \tau$, the state-space model is given by

$$\dot{x} = f(x) + g(x)u \quad (12.1)$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ c_1 c_3 \sin x_1 \\ -c_1 c_3 \sin x_1 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ -c_3 \\ \frac{c_2 c_3}{I_w} \end{pmatrix},$$

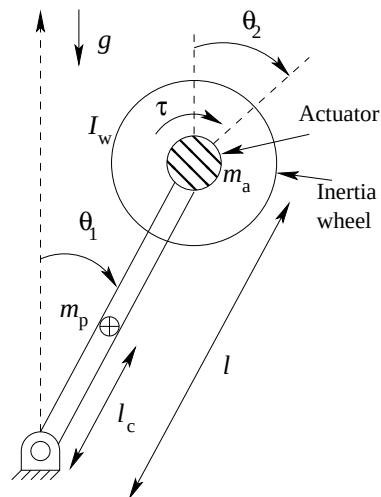


Figure 12.3: Schematic of the inertia wheel pendulum system

where, $c_1 \triangleq (m_p l_c + (m_a + m_w)l)g$, $c_2 \triangleq I_1 + I_w + (m_w + m_a)l^2$, $c_3 \triangleq \frac{I_w}{c_2 I_w - I_w^2}$, m_p is mass of the pendulum, l_c is center of mass of the pendulum, l is length of the pendulum, m_a is mass of the actuator, m_w is mass of the inertia wheel, I_1 is inertia of the pendulum about its end and I_w is inertia of the inertia wheel about the wheel center. The equilibrium set is given by $\mathcal{E} = \{x : (x_1, x_2, x_3, x_4) = (2n\pi, 2m\pi, 0, 0)\}$, $(n, m) \in \mathbb{Z}$.

Points to ponder

- Write the configuration space on which the Furuta pendulum evolves. How is it different from the configuration space of the pendulum on a cart system?

Lecture-13

Ball on a beam system

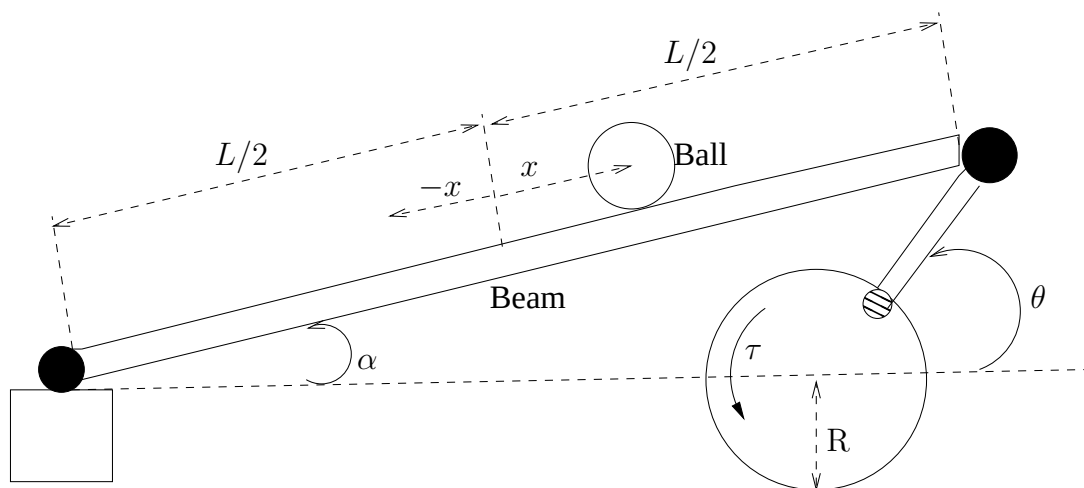


Figure 13.1: Ball and beam system

The system (see figure 13.1) consists of a ball that is free to move on a horizontal beam. The beam angle can be varied to control the position of the ball. Let α be the angle subtended by the beam with the horizontal and x denote the position of the ball measured with respect to the center of the beam. The connecting rod connects the gear to the right end of the beam (which is a rotary joint). Let θ be the angular distance covered by the gear, measured with respect to the horizontal. The ball is assumed to roll on the beam without slipping. It should also be noted that the approximation $R\theta = L\alpha$ holds for small angles.

Equations of Motion

The parameters of the system are depicted in table 13.1. The system has two degrees-

Parameter	Symbol
Length of the beam	L
Radius of the ball	r_b
Mass of the ball	M_b
Moment of inertia of the ball	J_b
Mass of the rod	M_r
Connecting rod displacement or radius of gear	R
Moment of inertia of the rod (about its pivot point)	J_r
Moment of inertia of the gear or flywheel	J_{fw}
Motor shaft torque	τ_m
Motor torque constant	K_t
Motor back-emf constant	K_m
Load torque	u_1
Motor input voltage	V_m
Motor current	I_m
Gear ratio	K_g
Gear efficiency	η_g
Motor armature resistance	R_m
Motor moment of inertia	J_m
Viscous damping coefficient on the load side	B_{eq}
Motor efficiency	η_m

Table 13.1: System parameters

of-freedom and is parameterized by the configuration variables (x, α) . The Lagrangian $\mathcal{L}(x, \alpha, \dot{x}, \dot{\alpha})$, defined as the difference between the kinetic and potential energy is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} J_r \dot{\alpha}^2 + \frac{1}{2} M_b \dot{x}^2 + \frac{1}{2} J_b \omega_b^2 + \frac{1}{2} M_b \left(\frac{L}{2} + x \right)^2 \dot{\alpha}^2 + \frac{1}{2} J_{fw} \dot{\theta}^2 \\ & - M_r g \frac{L}{2} \sin(\alpha) - M_b g \left(\frac{L}{2} + x \right) \sin(\alpha). \end{aligned}$$

The corresponding Euler-Lagrange equations of motion are given by

$$a_1 \ddot{\alpha} + a_2 \left(\frac{L}{2} + x \right)^2 \ddot{\alpha} + 2a_2 \left(\frac{L}{2} + x \right) \dot{\alpha} \dot{x} + a_3 \cos(\alpha) + a_2 g \left(\frac{L}{2} + x \right) \cos(\alpha) = u_1 - \left(\frac{B_{eq} L}{R} \right) \dot{\alpha} \quad (13.1)$$

and

$$\frac{7}{5} \ddot{x} - \left(\frac{L}{2} + x \right) \dot{\alpha}^2 + g \sin(\alpha) = 0 \quad (13.2)$$

where, the constants $\frac{J_r R}{L} + \frac{J_{fw} L}{R} = a_1$, $\frac{M_b R}{L} = a_2$, $\frac{M_r R g}{2} = a_3$. The torque provided by the motor is represented by u_1 .

Motor dynamics

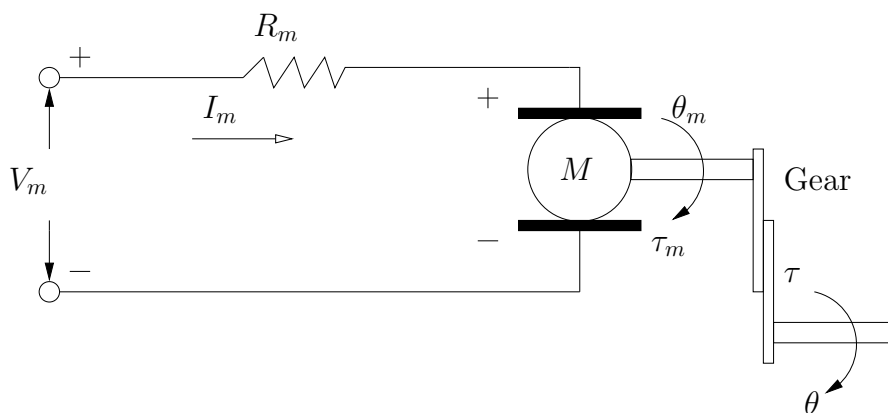


Figure 13.2: Motor circuit

The dynamics of the motor used to actuate the gear mechanism will be brought into the framework described by (13.1) and (13.2). This will enable one to apply a voltage to the motor as input rather than attempting to force the motor to provide the required torque u_1 . The dynamics of the motor is given by

$$\begin{aligned} J_m \ddot{\theta}_m &= \tau_m - \frac{u_1}{\eta_g K_g} \\ \tau_m &= \eta_m K_t I_m \\ I_m &= \frac{V_m - K_m \dot{\theta}_m}{R_m} \\ \theta_m &= K_g \theta. \end{aligned} \quad (13.3)$$

From (13.3) and by re-arranging terms, the following is obtained

$$u_1 = c_1 V_m - c_2 \dot{\alpha} - c_3 \ddot{\alpha} \quad (13.4)$$

where, $c_1 = \frac{\eta_g K_g \eta_m K_t}{R_m}$, $c_2 = \frac{\eta_g K_t K_m K_g^2 L}{R R_m}$ and $c_3 = \frac{L \eta_g K_g^2 J_m}{R}$. Equation (13.4) is substituted in (13.1) to yield

$$\begin{aligned} & a_1 \ddot{\alpha} + a_2 \left\{ \frac{L}{2} + x \right\}^2 \ddot{\alpha} + 2a_2 \left\{ \frac{L}{2} + x \right\} \dot{\alpha} \dot{x} + a_3 \cos(\alpha) \\ & + a_2 g \left\{ \frac{L}{2} + x \right\} \cos(\alpha) = c_1 V_m - c_2 \dot{\alpha} - c_3 \ddot{\alpha} - \left\{ \frac{B_{eq} L}{R} \right\} \dot{\alpha}. \end{aligned} \quad (13.5)$$

State-space formulation

The dynamics of the system given by (13.2) and (13.5) is expressed in the state-space form by choosing the state variables as $\mathbf{x} = (x_1, x_2, x_3, x_4) \triangleq (x, \dot{x}, \alpha, \dot{\alpha})$. The resulting state-space model is given by

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})V_m$$

where, the drift and control vector fields are given by

$$f = \begin{bmatrix} x_2 \\ \frac{5}{7} \left\{ \frac{L}{2} + x_1 \right\} x_4^2 - \frac{5}{7} g \sin(x_3) \\ x_4 \\ \frac{-\left\{ c_2 + \frac{B_{eq} L}{R} \right\} x_4 - 2a_2 \left\{ \frac{L}{2} + x_1 \right\} x_2 x_4 - \left\{ a_3 + a_2 g \left\{ \frac{L}{2} + x_1 \right\} \right\} \cos(x_3)}{a_1 + c_3 + a_2 \left\{ \frac{L}{2} + x_1 \right\}^2} \end{bmatrix}; g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{c_1}{a_1 + c_3 + a_2 \left\{ \frac{L}{2} + x_1 \right\}^2} \end{bmatrix}.$$

Points to ponder

- Does the system have an equilibrium point?
- Derive the equations of motion of a centrally pivoted ball-beam system? Does this configuration have an equilibrium point?

Lecture-14

Ball on a circular beam system

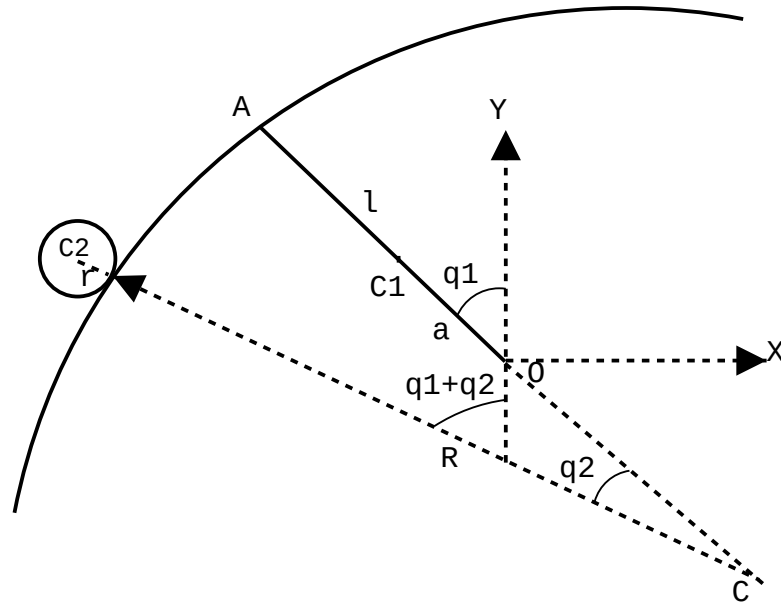


Figure 14.1: Schematic of circular beam and ball system

The circular beam and ball system consists of a circular beam with center C , radius R and a ball on it with center C_2 and radius r , as in Figure 14.1. The beam is actuated at the holder end O . C_1 is the center of mass of the beam with it's holder OA . Let m_1, ρ_1 and m_2, ρ_2 be the mass and radii of inertia of the beam with it's holder and ball respectively. Let l be the length of the holder, $OC_1 = a$ and g be the acceleration due to gravity. The kinetic energy is given by

$$K.E = \frac{m_1}{2} v_1^\top v_1 + \frac{m_2}{2} v_2^\top v_2$$

where,

$$\begin{aligned} v_1 &= \frac{d}{dt} \begin{pmatrix} -a \sin q_1 \\ a \cos q_1 \\ 0 \end{pmatrix} \\ v_2 &= \frac{d}{dt} \begin{pmatrix} -(R+r) \sin q_1 + q_2 + (R-l) \sin q_1 \\ (R+r) \cos q_1 + q_2 - (R-l) \cos q_1 \\ 0 \end{pmatrix} \end{aligned}$$

Defining the constants as,

$$\begin{aligned} a_1 &= m_1 \rho_1^2 + m_2 (R-l)^2 \\ a_2 &= m_2 (R+r)^2 \\ a_3 &= m_2 (R+r)(R-l) \\ a_4 &= m_2 (\rho_2 R/r)^2 \\ b_1 &= (m_1 a - m_2 (R-l))g \\ b_2 &= m_2 (R+r)g \end{aligned}$$

the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= (a_1 + a_2 - 2a_3 \cos q_2) \dot{q}_1^2 + (a_2 + a_4) \dot{q}_2^2 + (a_2 - a_3 \cos q_2) \dot{q}_1 \dot{q}_2 \\ &\quad - b_1 \cos q_1 + b_2 \cos (q_1 + q_2). \end{aligned}$$

Defining the states as $x = (x_1, x_2, x_3, x_4) \triangleq (q_1, q_2, \dot{q}_1, \dot{q}_2)$ the equations of motion can be expressed as

$$\dot{x} = f(x) + g(x)u$$

where the drift and control vector fields are given by,

$$f = \begin{pmatrix} x_3 \\ x_4 \\ f_3 \\ f_4 \end{pmatrix}; g = \begin{pmatrix} 0 \\ 0 \\ g_3 \\ g_4 \end{pmatrix}$$

with

$$\begin{aligned}
f_3 &= \frac{1}{\delta(x_2)} \{ -2a_2a_3x_3x_4 \sin x_2 - a_2a_3x_4^2 \sin x_2 + a_2b_1 \sin x_1 - 2a_4a_3x_3x_4 \sin x_2 \\
&\quad - a_4a_3x_4^2 \sin x_2 + a_4b_1 \sin x_1 + a_4b_2 \sin(x_1 + x_2) - a_2a_3x_3^2 \sin x_2 + a_3^2x_3^2 \cos x_2 \sin x_2 \\
&\quad + a_3 \cos x_2 b_2 \sin(x_1 + x_2) \} \\
f_4 &= \frac{1}{\delta(x_2)} \{ 2a_2a_3x_3x_4 \sin x_2 + a_2a_3x_4^2 \sin x_2 - a_2b_1 \sin x_1 - 2a_3^2x_3x_4 \cos x_2 \sin x_2 \\
&\quad - a_3^2x_4^2 \cos x_2 \sin x_2 + a_3 \cos x_2 b_1 \sin x_1 - a_3 \cos x_2 b_2 \sin(x_1 + x_2) + a_1a_3x_3^2 \sin x_2 \\
&\quad + a_1b_2 \sin(x_1 + x_2) + a_2a_3x_3^2 \sin x_2 - 2a_3^2x_3^2 \cos x_2 \sin x_2 \} \\
g_3 &= (a_2 + a_4)/\delta(x_2) \\
g_4 &= (-a_2 + a_3 \cos x_2)/\delta(x_2) \\
\delta(x_2) &= a_1a_2 + a_1a_4 + a_2a_4 - 2a_3a_4 \cos x_2 - a_3^2 \cos x_2^2
\end{aligned}$$

and $u = \tau$.

The cart on a beam system

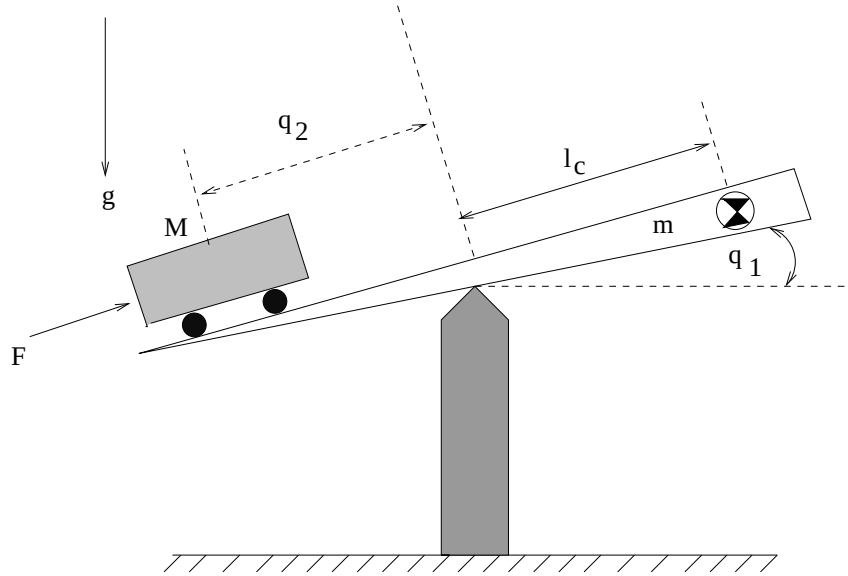


Figure 14.2: Schematic of the cart-beam system

The system consists of a beam of length l and mass m and the center of the beam is free to rotate on a pivot under the influence of gravity as shown in figure 14.2. The beam geometry is such that the center-of-mass is at a distance l_c from the center of the beam. A cart of mass M , actuated by a force F , traverses on the beam without slipping. The total energy of the CB system is given by

$$E(q, p) = \frac{1}{2} \dot{q}^\top D(q_2) \dot{q} + \left(q_2 + \frac{ml_c}{M} \right) M g \sin q_1 \quad (14.1)$$

where, $D(q) \triangleq \text{diag}\{J + Mq_2^2, M\}$, J denotes the moment-of-inertia of the beam about its pivot, q_1 is the angle made by the beam with the horizontal, $\dot{q}_1$ is the angular velocity, q_2 is the cart position, $\dot{q}_2$ is the cart velocity and g is the acceleration due to gravity. By choosing the states as $x \triangleq [x_1 \ x_2 \ x_3 \ x_4]^\top = [q_1 \ q_2 + \frac{ml_c}{M} \ \dot{q}_1 \ \dot{q}_2]^\top$, the Euler-Lagrange formulation of the CB system yields the following state-space representation

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{14.2}$$

where,

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{(-2Mx_3x_4\alpha(x_2) - Mgx_2 \cos x_1)}{\beta(x_2)} \\ x_3^2\alpha(x_2) - g \sin x_1 \end{pmatrix}; g(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where, $u \triangleq \frac{F}{M}$, $\alpha(x_2) \triangleq (x_2 - ml_c/M)$, $\beta(x_2) \triangleq J + M\alpha^2(x_2) > 0 \ \forall \ x_2$. If the beam angle is restricted to $(-\pi/2, \pi/2)$ to keep the cart in the upward configuration, then the system has a unique equilibrium point $x_e \triangleq (0, 0, 0, 0)$.

Points to ponder

- Linearize the system about the origin and find its eigen values. Compare the number of positive eigenvalues for the ball on a horizontal beam system with the ball on a circular beam system.
- Rewrite the dynamics for cart on a beam system when the center-of-mass of the beam coincides with the pivot point. Does the equilibrium point shift?

Lecture-15

Acrobot

The acrobot is a two-link manipulator that moves in a vertical plane under the influence of gravity. The manipulator is actuated by a single actuator at the elbow joint as shown in Figure 15.1. The equations of motion of the acrobot configuration space $Q = \mathbb{S}^1 \times \mathbb{S}^1$

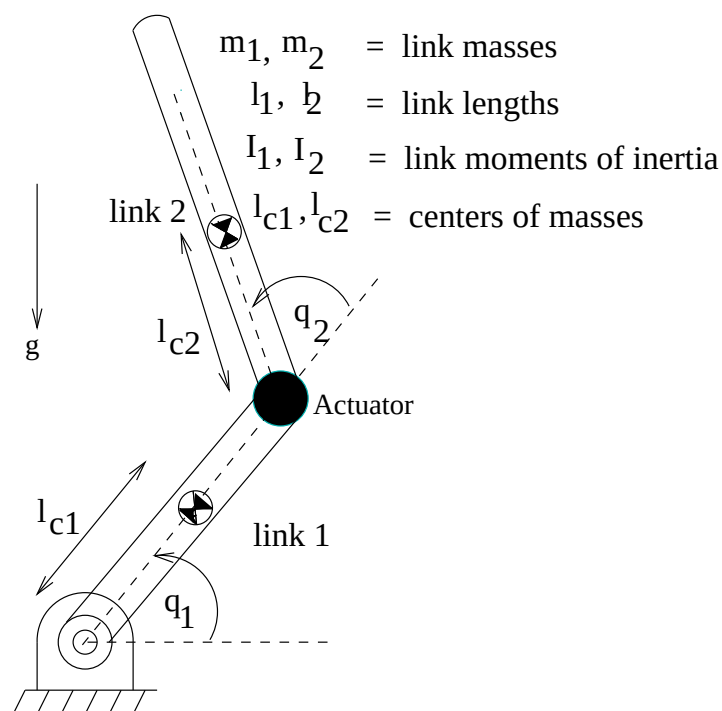


Figure 15.1: The Acrobot

are given by

$$\begin{aligned}
 d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \psi_1 &= 0 \\
 d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \psi_2 &= \tau_2
 \end{aligned}
 \tag{15.1}$$

where

$$\begin{aligned}
d_{11} &= c_1 + c_2 + 2c_3 \cos q_2 \\
d_{12} &= c_2 + c_3 \cos q_2 \\
d_{21} &= d_{12} \\
d_{22} &= c_2 \\
h_1 &= -c_3(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) \sin q_2 \\
h_2 &= c_3\dot{q}_1^2 \sin q_2 \\
\psi_1 &= c_4 g \cos q_1 + c_5 g \cos(q_1 + q_2) \\
\psi_2 &= c_5 g \cos(q_1 + q_2).
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\
c_2 &= m_2 l_{c2}^2 + I_2 \\
c_3 &= m_2 l_1 l_{c2} \\
c_4 &= m_1 l_{c1} + m_2 l_1 \\
c_5 &= m_2 l_{c2}.
\end{aligned}$$

Equation (15.1) can be rewritten as Defining the state vector as $x = (x_1, x_2, x_3, x_4) \triangleq ((q_1 - \frac{\pi}{2}), q_2, \dot{q}_1, \dot{q}_2)$, the dynamics can be represented as

$$\dot{x} = f(x) + g(x)u \quad (15.2)$$

where the state-space manifold is $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$ and the smooth vector fields f and g are given by

$$f = \begin{pmatrix} x_3 \\ x_4 \\ f_3 \\ f_4 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 0 \\ \frac{-(c_2 + c_3 \cos x_2)}{\gamma(x_2)} \\ \frac{(c_1 + c_2 + 2c_3 \cos x_2)}{\gamma(x_2)} \end{pmatrix}$$

where

$$\begin{aligned}
f_3 &= \frac{1}{\gamma(x_2)} \{ c_2 c_3 (x_3 + x_4)^2 \sin x_2 + c_2 c_4 g \sin x_1 + c_3^2 x_3^2 \cos x_2 \sin x_2 \\
&\quad - c_3 c_5 g \sin(x_1 + x_2) \cos x_2 \} \\
f_4 &= \frac{1}{\gamma(x_2)} \{ -(c_2 + c_3 \cos x_2)(2x_3 x_4 + x_4^2) c_3 \sin x_2 - c_2 c_4 g \sin x_1 \\
&\quad - c_3 c_4 g \cos x_2 \sin x_1 + c_1 c_5 g \sin(x_1 + x_2) + c_3 c_5 g \cos x_2 \sin(x_1 + x_2) \\
&\quad - (c_1 + c_2 + 2c_3 \cos x_2) x_3^2 c_3 \sin x_2 \} \\
\gamma(x_2) &= c_1 c_2 - c_3^2 \cos^2 x_2
\end{aligned}$$

and $u = \tau_2$. The system has four equilibrium points given by

$$\{x_{ij}^e \in M : x_1 = -i\pi, x_2 = j\pi, x_3 = 0, x_4 = 0\} \quad i, j = 0, 1.$$

A variant of the acrobot is the pendubot, the difference being that the pendubot is actuated only at the shoulder joint.

PPR manipulator

This manipulator has two prismatic joints which are actuated, while the third joint is a revolute passive joint, hence the name PPR. The two prismatic joints are orthogonal. Let (x, y) denote the position of the free joint and θ the orientation of the third link. It is clear that the configuration space of a PPR manipulator is $Q = \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ and is parameterized by the coordinates $q = [x \ y \ \theta]$. Let F_1 and F_2 be the control inputs to the two prismatic joints. Also, denote by l the distance between the center of mass of the third link and the third joint. The equations of motion using the Newtonian approach are

$$\begin{aligned}
m_x \ddot{x} - m_3 l \dot{\theta}^2 \cos \theta - m_3 l \ddot{\theta} \sin \theta &= F_1 \\
m_y \ddot{y} - m_3 l \dot{\theta}^2 \sin \theta + m_3 l \ddot{\theta} \cos \theta &= F_2 \\
I \ddot{\theta} + m_3 l \ddot{y} \cos \theta - m_3 l \ddot{x} \sin \theta &= 0,
\end{aligned} \tag{15.3}$$

where $m_x = m_1 + m_2 + m_3$, $m_y = m_2 + m_3$ and $I = I_3 + m_3 l^2$. Clearly all points $q \in Q$ constitute the equilibrium configuration at zero control inputs.

Points to ponder

- Identify the Coriolis and centripetal forces?

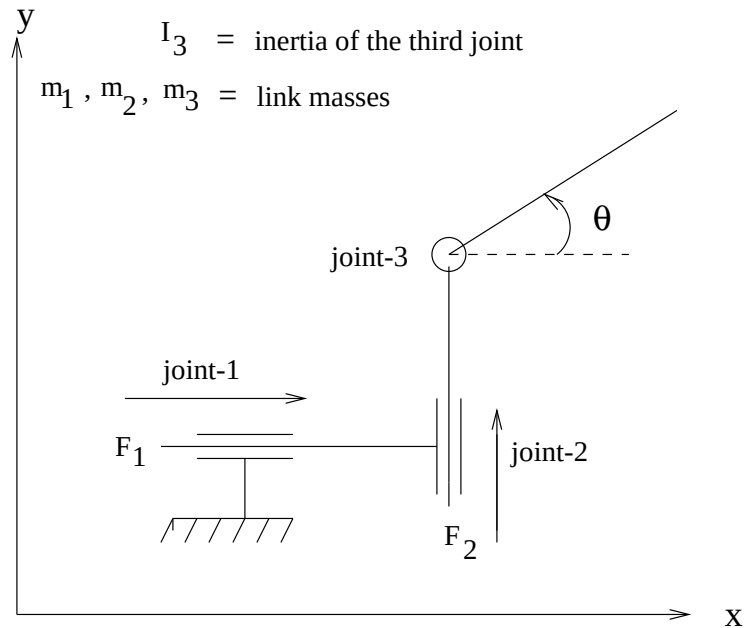


Figure 15.2: PPR planar manipulator

- Intuitively, which of the equilibrium points is stable?
- Write the input vector G for the pendubot?
- Linearize the PPR system at the origin. Check for controllability of the linearized system using Kalman's rank condition. You will observe that the rank condition fails and hence the system is not completely controllable. What is the cause for the loss of controllability for the linearized system?

Lecture-16

Mobile robot

The mobile robot is a popular robotic system that finds application in a multitude of tasks and is also a challenging problem from a control point-of-view. The kinematic model of the mobile robot as shown in figure 16.1 can be expressed as

$$\begin{aligned}\dot{x} &= v \cos \theta \\ \dot{y} &= v \sin \theta \\ \dot{\theta} &= \omega\end{aligned}\tag{16.1}$$

where, the triple (x, y, θ) denotes the position and the orientation of the vehicle with respect to the inertial frame and v, ω are the linear and angular velocities of the mobile robot. The dynamic model is obtained using the following relations.

$$\begin{aligned}M\dot{v} &= F \\ I\dot{\omega} &= \tau\end{aligned}\tag{16.2}$$

where, M is the mass of the vehicle, I is moment-of-inertia, $\tau = \frac{L}{r}(\tau_1 - \tau_2)$, $F = \frac{1}{r}(\tau_1 + \tau_2)$, with L being the distance between the center-of-mass and the wheel, τ_1, τ_2 , the left and right wheel motor torques and r is the radius of the rear wheel. Two assumptions are made to simplify the model: the center-of-mass and the rear-axis center coincide and the wheels do not slide. The second assumption leads to a velocity level nonholonomic constraint $\dot{x} \sin \theta - \dot{y} \cos \theta = 0$. With the following choice of state-vector

$$\begin{aligned}x_1 &= \theta \\ x_2 &= x \cos \theta + y \sin \theta \\ x_3 &= (\theta y - 2x) \sin \theta + (2y + \theta x) \cos \theta \\ x_4 &= \omega \\ x_5 &= v - \omega(x \sin \theta - y \cos \theta),\end{aligned}$$

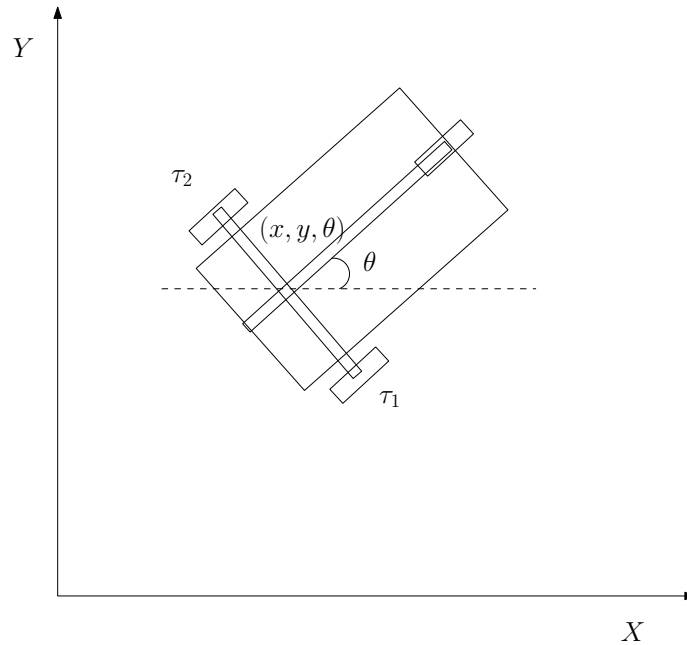


Figure 16.1: Mobile robot

equations (16.1) - (16.2) can be combined to yield a state-space representation on the manifold $S^1 \times \mathbb{R}^4$ as

$$\dot{x} = f(x) + g_1(x_1)\tau + g_2(x)F \quad (16.3)$$

where

$$f(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_1x_5 - x_2x_4 \\ 0 \\ -x_4^2x_2 \end{pmatrix}; g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ k_1 \\ -\frac{k_1(x_1x_2 - x_3)}{2} \end{pmatrix}; g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_2 \end{pmatrix}, k_1 \triangleq \frac{1}{I}, k_2 \triangleq \frac{1}{M}.$$

Points to ponder

- Rewrite the kinematic model of the mobile robot assuming that the robot has a differential drive. The model should have the wheel angular velocities (ω_r, ω_l) as control inputs.
- For the differential drive model derived, analyze the motion of the robot for the following control inputs. a.) When $\omega_r = -\omega_l$, and $|\omega_r|$ is a constant.
b.) When $\omega_r - \omega_l = 0$, and $|\omega_r|$ and $|\omega_l|$ are constants, but unequal.

Lecture-17

Bioreactor Model

A bioreactor is a tank in which several biological reactions occur simultaneously in a liquid medium. In the simplest reactor two components are considered, biomass and substrate. Biomass consists of cells that consume the substrate. Consider the schematic of a biochemical reactor in Figure 17.1, where X and S are the cell and substrate concentrations, F is the feed flow rate of substrate (assuming no biomass in feed) and V the volume of the reactor. The continuous time representation of the bioreactor system is:

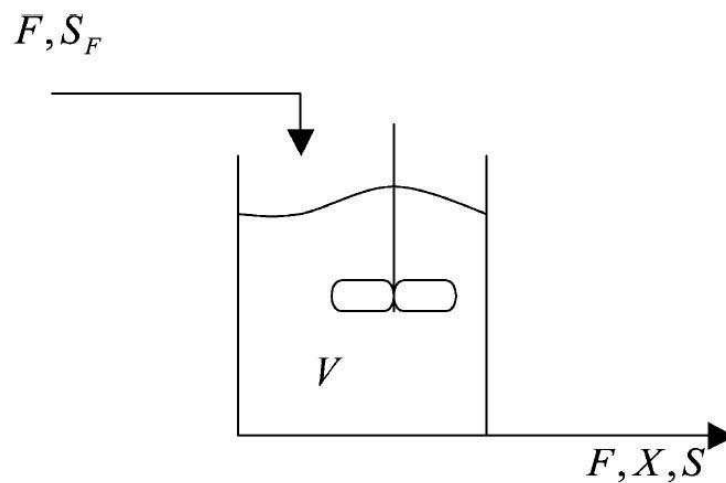


Figure 17.1: Bioreactor

$$\begin{aligned}\frac{dX}{dt} &= -\frac{F}{V}X + \mu(S) \\ \frac{dS}{dt} &= -\frac{F}{V}(S_F - S) - \sigma(S)X\end{aligned}\tag{17.1}$$

where S is the substrate concentration and X is the cell concentration. The growth model where all of the biomass cells are “lumped together” in a single (one-hump) model

$$\mu(S) = KSe^{-\frac{S}{K}} \quad (17.2)$$

is used to represent the substrate growth rate as a function of biomass with K as the growth model coefficient. The specific substrate consumption rate, σ , is related to the yield coefficient Y in the form

$$Y(S) = \frac{\mu(S)}{\sigma(S)} = a + bS \quad (17.3)$$

where a and b are positive constants used to approximate the increasing portion of the yield curve. Combining (17.1)-(17.3), the dimensionless cell and substrate mass balances can be represented as

$$\begin{aligned} \frac{dC_1}{d\tau} &= -C_1 + Da(1 - C_2)e^{\frac{C_2}{\gamma}} C_1 \\ \frac{dC_2}{d\tau} &= -C_2 + Da\frac{1+\beta}{1+\beta-C_2}(1 - C_2)e^{\frac{C_2}{\gamma}} C_1 \end{aligned} \quad (17.4)$$

where C_1 and C_2 are the dimensionless cell mass and substrate conversion, respectively. Brengel and Seider transformed (17.4) to yield the following system of equations incorporating the manipulated input variable, u which is the dimensionless feed flow rate.

$$\begin{aligned} \frac{dX_1}{d\tau_S} &= -(C_1S + X_1)(1 + u) + Da_S(1 - X_2 - C_2S)e^{\frac{(C_2S + X_2)}{\gamma}}(C_1S + X_1) \\ \frac{dX_2}{d\tau_S} &= -(C_2S + X_2)(1 + u) + Da_S(1 - X_2 - C_2S)e^{\frac{(C_2S + X_2)}{\gamma}}\frac{1+\beta}{1+\beta-X_2-C_2S}(C_1S + X_1) \end{aligned} \quad (17.5)$$

where τ_S represents dimensionless time under steady state conditions Da_S , the Damkohler number at the steady state, X_1 the dimensionless cell mass deviation from steady state and X_2 is the corresponding dimensionless substrate conversion deviation.

DC motor

The traditional model of DC motor is a second-order linear one, which ignores the dead nonlinear zone of the motor. Unfortunately, the dead zone caused by the nonlinear friction would bring great effect to servo systems. Therefore, it is vital to model the dead zone of DC motor accurately in order to improve the performance of servo system. In order to simplify applications and reflect the real nonlinear friction of the motor accurately, a simplified friction model was proposed by Cong S. et al. , which is expressed as

$$T_f = T_c \text{sgn}(\omega) + (T_s - T_c)e^{(-\alpha|\omega|)} \text{sgn}(\omega) \quad (17.6)$$

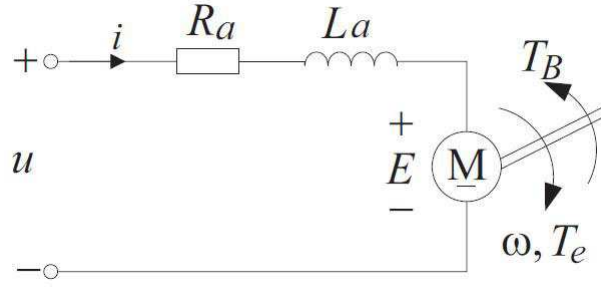


Figure 17.2: DC motor

Where, T_c is the Coulomb friction torque (Nm); T_s is static friction torque (Nm); α is time constant; ω is the angular speed of the rotor (rad/s). The DC motor equivalent circuit under rating excitation is as shown in Figure 17.2. The voltage balance equation of DC motor armature circuit is expressed as

$$u = R_a i + L_a \frac{di}{dt} + K_e \omega$$

where, i is armature current (A); L_a is equivalent inductance of armature circuit (H); R_a is equivalent resistance of armature circuit (Ω); u is terminal voltage of armature circuit (V); K_e is voltage coefficient of DC motor (Vs/rad). The torque balance equation of DC motor is expressed as

$$K_t i - B\omega = J \frac{d\omega}{dt}$$

where, J is the inertia moment of the rotor (Kgm^2); K_t is the torque coefficient of DC motor (Nm/A). B is viscous friction coefficient of DC motor (Nms/rad). The linear DC motor model is obtained as

$$\begin{aligned} J\dot{\omega} &= -B\omega + K_t i \\ L_a \dot{i} &= -K_e \omega - R_a i + u \end{aligned}$$

Considering the effect of nonlinear friction, the nonlinear friction torque is applied to model the dead zone of DC motor. Then the nonlinear DC motor model is shown as

$$\begin{aligned} J\dot{\omega} &= -B\omega + K_t i - T_c \text{sgn}(\omega) - (T_s - T_c) e^{(-\alpha|\omega|)} \text{sgn}(\omega) \\ L_a \dot{i} &= -K_e \omega - R_a i + u \end{aligned} \quad (17.7)$$

The DC motor model is transformed into second-order nonlinear model from traditional second-order linear one due to the presence of nonlinear friction torque, which makes the model of DC motor more accurate.

Exercise problems

1. Using suitable states, represent the Furuta pendulum dynamics in state-space form and find the equilibrium set.
2. Find the equilibrium set for the ball on a circular beam system.
3. The dynamics of a DC motor can be described by

$$\begin{aligned}v_f &= R_f i_f + L_f \dot{i}_f \\v_a &= c_1 i_f \omega + L_a \dot{i}_a + R_a i_a \\J\dot{\omega} &= c_2 i_f i_a - c_3 \omega\end{aligned}$$

The first equation is for the field circuit with v_f , R_f and L_f being its voltage, current, resistance and inductance. The variables v_a , R_a and L_a are the corresponding variables for the armature circuit described by the second equation. The third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as a damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

- a. For a separately excited D.C motor, the voltages v_a and v_f are independent control inputs. Choose appropriate state variables and find the state equation.
- b. Specialize the state equation of part (a) to the field controlled DC motor, where v_f is the control input while v_a is held constant.
- c. Specialize the state equation of part (a) to the armature controlled DC motor, where v_a is the control input while v_f is held constant. Can you reduce the order of the model in this case?

- d. In a shunt wound DC motor, the field and armature windings are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux; that is, $v = v_a = v_f + R_x i_f$. With v as the control input, write down the state equation.

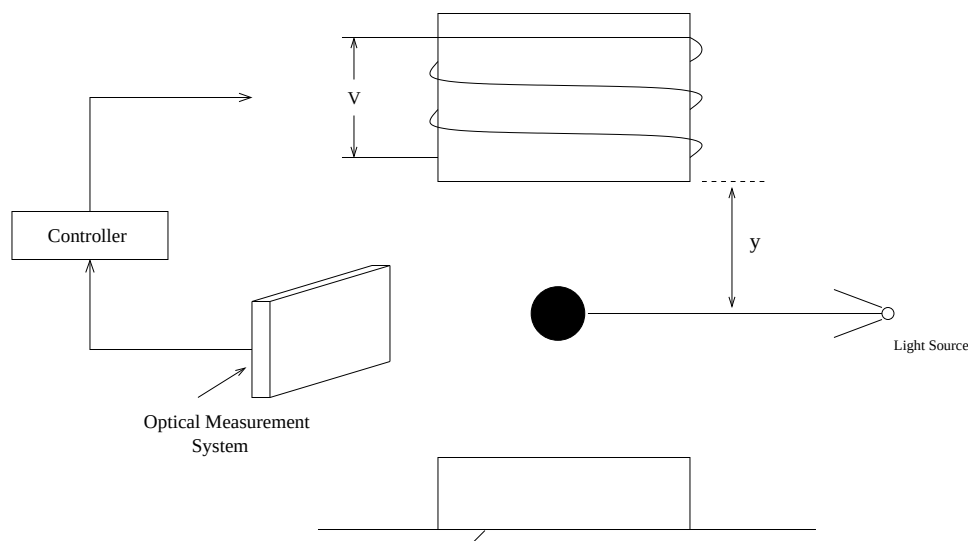


Figure 17.3: Magnetic suspension system

4. The schematic diagram of a magnetic suspension system (see Figure (17.3)), where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the optically measured ball position. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers and fast trains. The equation of motion of the ball is $m\ddot{y} = -k\dot{y} + mg + F(y, i)$, where m is the mass of the ball, $y \geq 0$ is the vertical (downward) position of the ball measured from a reference point ($y = 0$ when the ball is next to the coil), k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y, i)$ is the force generated by the electromagnet, and i is its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as $L(y) = L_1 + L_0/(1 + y/a)$ where L_1, L_0 and a are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y, i) = 0.5L(y)i^2$ as the energy stored in the electromagnet, the force $F(y, i)$ is given by $F(y, i) = -\frac{L_0 i^2}{(2a(1+y/a)^2)}$. When the electric circuit of the coil is driven by a voltage source with voltage v , Kirchoff's voltage law gives the relationship $v = \dot{\phi} + Ri$, where R is the series resistance of the circuit and $\phi = L(y)i$

is the magnetic flux linkage.

- a. Using $x_1 = y, x_2 = \dot{y}, x_3 = i$ as state variables and $u = v$ as control input, find the state equation.
 - b. Suppose it is desired to balance the ball at a certain position $r > 0$. Find the steady state values I_{ss} and V_{ss} of i and v , respectively, which are necessary to maintain such balance.
5. Consider a magnetic beam balance (MBB) system, that consists of an off-centered beam of length l free to rotate on a pivot, under the influence of gravity, as shown in Figure 17.4.

The beam geometry is such that the center-of-mass is at a distance l_c from the center of the beam. An electromagnet A is placed below one end of the beam and a dummy electromagnet B is placed at the other end to incorporate physical constraints into the system and obtain a symmetrical arrangement.

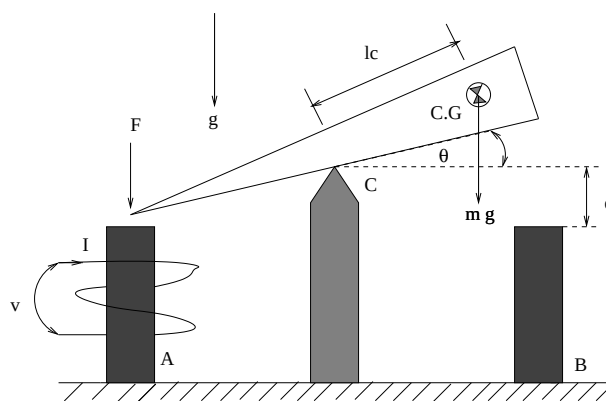


Figure 17.4: Schematic of the beam-balance system

Let J, m, l_c denote the moment-of-inertia, mass and distance between the pivot and center-of-mass of the beam, θ is the angle made by the beam with the horizontal, $\dot{\theta}$ is the angular velocity, g is the acceleration due to gravity and F is the force created by the electromagnet. Further, let l denote the length of the beam and e the air-gap. The electromagnetic circuit is given by

$$v = L\dot{I} + I\dot{L} + IR \quad (17.8)$$

where R and L are the resistance and inductance of the coil respectively, I is the current and v is the voltage applied across the coil terminals. The inductance L is

related to the beam angle as $L = \frac{L_0 \theta_m}{\theta_m + \theta}$, where L_0 is the inductance of the coil when $\theta = 0$ and θ_m is the maximal angle that is reached when one end of the beam touches the electromagnet and is given by $\theta_m = \arctan(2e/l)$. Finally the expression for the force F is given by

$$F = \lambda \left(\frac{\theta_m I}{\theta_m + \theta} \right)^2 \quad (17.9)$$

where λ is a positive constant. In arriving at the above equations, it is assumed that the permeability of the magnetic material is constant over the operating range and the flux density is uniformly distributed across the air-gap. Also, the effect of eddy current, stray flux and flux leakage are assumed to be small that they can be neglected. Find the state-space representation of the system with a) with F as the control input , (b) with the voltage v as the current input.

6. What is the number of isolated equilibrium points for an n -link manipulator moving in a vertical plane under the influence of gravity?