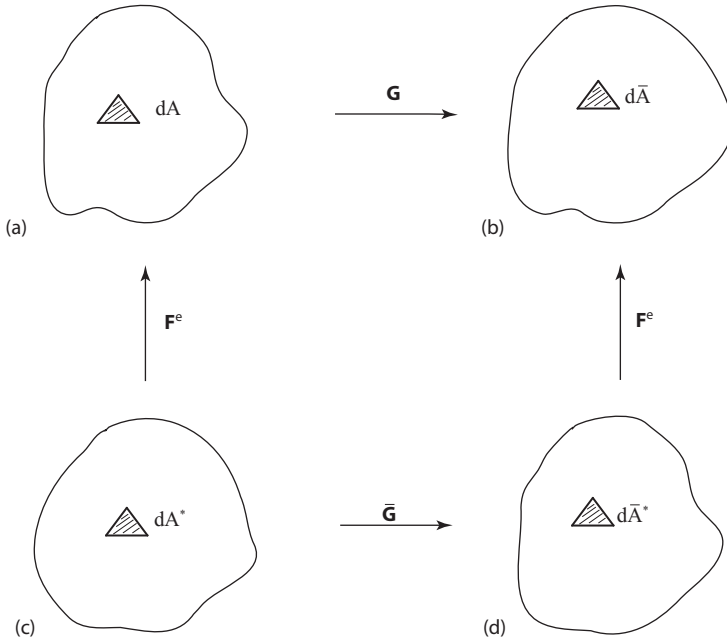

Coupled Damage Plasticity

A constitutive model is developed in this chapter for anisotropic continuum damage mechanics using finite-strain plasticity. The formulation is given in spatial coordinates (Eulerian reference frame) and incorporates both isotropic and kinematic hardening. The von Mises yield function is modified to include the effects of damage through the use of the hypothesis of elastic energy equivalence. A modified elasto-plastic stiffness tensor that includes the effects of damage is derived within the framework of the proposed model.

It is also shown how the model can be used in conjunction with other damage-related yield criteria. In particular, Gurson's yield function (Gurson 1977) which was later modified by Tvergaard (1982) and Tvergaard and Needleman (1984) is incorporated in the proposed theory. This yield function is derived based on the presence of spherical voids in the material and an evolution law for the void growth is also incorporated. It also shows how a modified Gurson yield function can be related to the proposed model. Some interesting results are obtained in this case. For more details, the reader is referred to Kattan and Voyiadjis (1990, 1993, 2001 a,b), Krajcinovic (1983, 1984), Voyiadjis and Kattan (1996), Voyiadjis and Park (1999), and Wilt and Arnold (1994).

7.1 Stress Transformation between Damaged and Undamaged States

Consider a body in the initial undeformed and undamaged configuration C_o . Let C be the configuration of the body that is both deformed and damaged after a set of external agencies act on it. Next, consider a fictitious configuration of the body $\bar{C}$ obtained from C by removing all the damage that the body has undergone. In other words, $\bar{C}$ is the state of the body after it had only deformed without damage. Therefore, in defining a damage tensor ϕ , its components must vanish in the configuration $\bar{C}$ (see [Figure 7.1](#))

**FIGURE 7.1**

States of deformation and damage: (a) deformed damage state, (b) fictitious deformed undamaged state, (c) elastically unloaded damaged state (unstressed state), (d) elastically unloaded fictitious undamaged state (fictitious unstressed undamaged state). (Voyiadjis and Kattan, 1999)

7.1.1 Effective Stress Tensor

In the formulation that follows, the Eulerian reference system is used, i.e., all the actual quantities are referred to the configuration C while the effective quantities referred to $\bar{C}$. One first introduces a linear transformation between the Cauchy stress tensor σ and the effective Cauchy stress tensor $\bar{\sigma}$ in the following form:

$$\bar{\sigma}_{ij} = M_{ijkl} \sigma_{kl} \quad (7.1)$$

where M_{ijkl} are the components of the fourth-order linear operator called the damage effect tensor.

A transformation relation for the deviatoric Cauchy stress tensor is next derived. One first writes the deviatoric part σ' in the configuration C as follows:

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{mm}\delta_{ij} \quad (7.2)$$

where δ_{ij} are the components of the second-order identity tensor $\mathbf{I}$. A similar relation exists in the configuration $\bar{C}$ between $\bar{\sigma}$ and $\bar{\sigma}'$ in the following form:

$$\bar{\sigma}'_{ij} = \bar{\sigma}_{ij} - \frac{1}{3}\bar{\sigma}_{mm}\delta_{ij} \quad (7.3)$$

where δ_{ij} is the same in both C and $\bar{C}$. Substituting for $\bar{\sigma}_{ij}$ from equation (7.1) into equation (7.3) while using equation (7.2), we obtain the following:

$$\bar{\sigma}'_{ij} = M_{ijkl}\sigma'_{kl} + \frac{1}{3}M_{ijmm}\sigma_{nn} - \frac{1}{3}M_{ppqr}\sigma_{qr}\delta_{ij} \quad (7.4)$$

It is clear from equation (7.4) that a linear relation does not exist between $\bar{\sigma}'$ and σ' . On the other hand, one might suspect that the last two terms on the right-hand side of the equation (7.4) cancel each other when they are written in expanded form. However, this possibility can be easily dismissed as follows: suppose one assumes $\bar{\sigma}'_{ij} = M_{ijkl}\sigma'_{kl}$. Using this with equation (7.4) one concludes that $M_{ijkl}\sigma_{nn} = M_{ppqr}\sigma_{qr}\delta_{ij}$. Now consider the case when $i \neq j$. One has $\delta_{ij} = 0$ and, therefore, $M_{ijmm}\sigma_{nn} = 0$. It is clear that this is a contradiction to the fact that generally $M_{ijmm} \neq 0$ and $\sigma_{nn} \neq 0$. Therefore, the additional terms in the equation (7.4) are non-trivial and such a linear transformation cannot be assumed.

Upon examining equation (7.4) in more detail, eliminating σ'_{kl} by using equation (7.2) and simplifying the resulting expression, one obtains the following:

$$\bar{\sigma}'_{ij} = N_{ijkl}\sigma_{kl} \quad (7.5)$$

where N_{ijkl} are the components of a fourth-order tensor given by:

$$N_{ijkl} = M_{ijkl} - \frac{1}{3}M_{rrkl}\delta_{ij} \quad (7.6)$$

Equation (7.5) represents a linear transformation between the effective deviatoric Cauchy stress tensor $\bar{\sigma}'$ and the Cauchy stress tensor σ . However, in this case the linear operator $\mathbf{N}$ is not simply the damage effect tensor $\mathbf{M}$ but a linear function of $\mathbf{M}$ as shown in equation (7.6). The tensors $\mathbf{M}$ and $\mathbf{N}$ are mappings $S \rightarrow \bar{S}$ and $S \rightarrow \bar{S}_{dev.}$, respectively, where S is the stress space in the current configuration C and $\bar{S}$ is the stress space in the fictitious undamaged state $\bar{C}$, with $\sigma \in S$ and $\bar{\sigma} \in \bar{S}$.

Next, we consider the effective stress invariants and their transformation in the configuration $\bar{C}$. It is seen from equation (7.5) that the first effective deviatoric stress invariant $\bar{\sigma}'_{ii}$ is given by:

$$\bar{\sigma}'_{ii} = N_{iikl}\sigma_{kl} = 0 \quad (7.7)$$

since $N_{iikl} = 0$ by direct contraction in equation (7.6). Therefore, one obtains $\bar{\sigma}'_{ii} = \sigma'_{ii} = 0$.

The problem becomes more involved when considering the effective stress invariant $\bar{\sigma}'_{ij}\bar{\sigma}'_{ij}$. Using equation (7.4) along with equation (7.2), one obtains:

$$\bar{\sigma}'_{ij}\bar{\sigma}'_{ij} = A_{klmn}\sigma'_{kl}\sigma'_{mn} + B_{pq}\sigma'_{pq} + C \quad (7.8)$$

where

$$A_{klmn} = M_{ijkl}M_{ijmn} - \frac{1}{3}M_{ttmn}M_{rrkl} \quad (7.9a)$$

$$B_{pq} = \frac{2}{3}\sigma_{mm}(M_{ijkk}M_{ijpq} - \frac{1}{3}M_{ttpq}M_{rrnn}) \quad (7.9b)$$

$$C = \frac{1}{2}\sigma_{mm}^2(M_{ijpp}M_{ijqq} - \frac{1}{3}M_{ttmm}M_{rrnn}) \quad (7.9c)$$

Substituting for σ' from equation (7.2) into equation (7.8) [or more directly using equation (7.5) along with equation (7.6)], one obtains:

$$\bar{\sigma}'_{ij}\bar{\sigma}'_{ij} = H_{klmn}\sigma_{kl}\sigma_{mn} \quad (7.10)$$

where the fourth-order tensor $\mathbf{H}$ is given by:

$$H_{klmn} = N_{ijkl}N_{ijmn} \quad (7.11)$$

and the tensor $\mathbf{N}$ is given by equation (7.6). The transformation equation (7.10) will be used in the next sections to transform the von Mises yield criterion into the configuration $\bar{C}$.

7.1.2 Effective Backstress Tensor

In the theory of plasticity, kinematic hardening is modeled by the motion of the yield surface in the stress space. This is implemented mathematically by the evolution of the shift or backstress tensor $\mathbf{X}$. The backstress tensor $\mathbf{X}$ denotes the position of the center of the yield surface in the stress space. For this purpose, one studies now the transformation of this tensor in the configuration C and $\bar{C}$.

Let $\mathbf{X}'$ be the deviatoric part of $\mathbf{X}$. Therefore, one has the following relation:

$$X'_{ij} = X_{ij} - \frac{1}{3}X_{mm}\delta_{ij} \quad (7.12)$$

where both $\mathbf{X}$ and $\mathbf{X}'$ are referred to the configuration C . Let their effective counterparts $\bar{\mathbf{X}}$ and $\bar{\mathbf{X}}'$ be referred to the configuration $\bar{C}$. Similarly to equation (7.12), we have:

$$\bar{X}'_{ij} = \bar{X}_{ij} - \frac{1}{3}\bar{X}_{mm}\delta_{ij} \quad (7.13)$$

Assuming a linear transformation (based on the same argument used for the stresses) similar to equation (7.1) between the effective backstress tensor $\bar{\mathbf{X}}$ and the backstress tensor $\mathbf{X}$:

$$\bar{X}_{ij} = M_{ijkl}X_{kl} \quad (7.14)$$

and following the same procedure in the derivation of equation (7.5), we obtain the following linear transformation between $\mathbf{X}$ and $\bar{\mathbf{X}}'$:

$$\bar{X}'_{ij} = N_{ijkl}X_{kl} \quad (7.15)$$

The effective backstress invariants have similar forms to those of the effective stress invariants, mainly, $\bar{X}'_{ii} = X'_{ii} = 0$, and

$$\bar{X}'_{ij}\bar{X}'_{ij} = H_{klmn}X_{kl}X_{mn} \quad (7.16)$$

In addition, one more transformation equation needs to be given before we can proceed to the constitutive model. By following the same procedure for the other invariants, the mixed invariant $\sigma_{ij}X_{ij}$ in the configuration C is transformed to $\bar{\sigma}'_{ij}\bar{X}'_{ij}$ as follows:

$$\bar{\sigma}'_{ij}\bar{X}'_{ij} = H_{klmn}\sigma_{kl}X_{mn} \quad (7.17)$$

and a similar relation holds for the invariant $\bar{X}'_{ij}\bar{\sigma}'_{ij}$. The stress and backstress transformation equations will be used later in the constitutive model.

7.2 Strain Rate Transformation between Damaged and Undamaged States

In the general elasto-plastic analysis of deforming bodies, the spatial strain rate tensor $\mathbf{d}$ in the configuration C is decomposed additively as follows (Nemat-Nasser, 1979, 1983; Lee, 1981):

$$d_{ke} = d'_{kl} + d''_{kl} \quad (7.18)$$

where $\mathbf{d}'$ and $\mathbf{d}''$ denote the elastic and plastic parts of $\mathbf{d}$, respectively. In equation (7.18), the assumption of small elastic strains is made, however,

finite plastic deformations are allowed. On the other hand, the decomposition in equation (7.18) will be true for any amount of elastic strain if the physics of elasto-plasticity is invoked, for example the case of single crystals. A thorough account of this is given by Asaro (1983).

In the next two subsections, the necessary transformation equations between the configurations C and $\bar{C}$ will be derived for the elastic strain and plastic strain rate tensors. In this derivation, it is assumed that the elastic strains are small compared with the plastic strains and, consequently, the elastic strain tensor is taken to be the usual engineering elastic strain tensor ϵ' . In addition, it is assumed that an elastic strain energy function exists such that a linear relation can be used between the Cauchy stress tensor σ and the engineering elastic strain tensor ϵ' . The tensor ϵ' is defined here as the linear term of the elastic part of the spatial strain tensor where second-order terms are neglected. For more details, see the work by Kattan and Voyiadjis (1992) and Voyiadjis and Kattan (1999).

7.2.1 Effective Elastic Strain

The elastic constitutive equation to be used is based on one of the assumptions outlined in the previous paragraph and is represented by the following linear relation in the configuration $\bar{C}$:

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl} \bar{\epsilon}'_{kl} \quad (7.19)$$

where $\bar{E}$ is the fourth-rank elasticity tensor given by:

$$\bar{E}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (7.20)$$

where λ and M are Lamé's constants. Based on the constitutive equation (7.19), the elastic strain energy function $U(\epsilon', \phi)$ in the configuration $\bar{C}$ is given by:

$$U(\epsilon', \mathbf{0}) = \frac{1}{2} \bar{E}_{ijkl} \bar{\epsilon}'_{ij} \bar{\epsilon}'_{kl} \quad (7.21)$$

One can now define the complementary elastic energy function $V(\sigma, \phi)$, based on a Legendre transformation, as follows:

$$V(\sigma, \phi) = \sigma_{ij} \epsilon'_{ij} - U(\bar{\epsilon}', \phi) \quad (7.22)$$

By taking the partial derivative of equation (7.22) with respect to the stress tensor σ , one obtains:

$$\epsilon'_{ij} = \frac{\partial V(\sigma, \phi)}{\partial \sigma_{ij}} \quad (7.23)$$

Substituting expression (7.21) into equation (7.22) in the configuration $\bar{C}$, one obtains the following expression for $V(\boldsymbol{\sigma}, \boldsymbol{\phi})$ in the configuration $\bar{C}$ as follows:

$$V(\bar{\boldsymbol{\sigma}}, \mathbf{0}) = \frac{1}{2} \bar{E}_{ijkl}^{-1} \bar{\sigma}_{ij} \bar{\sigma}_{kl} \quad (7.24)$$

The hypothesis of elastic energy equivalence, which was initially proposed by Sidoroff (1981), is now used to obtain the required relation between $\boldsymbol{\epsilon}'$ and $\bar{\boldsymbol{\epsilon}}'$. In this hypothesis, one assumes that the elastic energy $V(\boldsymbol{\sigma}, \boldsymbol{\phi})$ in the configuration C is equivalent in form to $V(\bar{\boldsymbol{\sigma}}, \mathbf{0})$ in the configuration $\bar{C}$. Therefore, one writes:

$$V(\boldsymbol{\sigma}, \boldsymbol{\phi}) = V(\bar{\boldsymbol{\sigma}}, \mathbf{0}) \quad (7.25)$$

where $V(\boldsymbol{\sigma}, \boldsymbol{\phi})$ is the complementary elastic energy in C and is given by:

$$V(\boldsymbol{\sigma}, \boldsymbol{\phi}) = \frac{1}{2} E_{ijkl}^{-1}(\boldsymbol{\phi}) \sigma_{ij} \sigma_{kl} \quad (7.26)$$

where the superscript -1 indicates the inverse of the tensor.

In the equation (7.26), the effective elasticity modulus $\mathbf{E}(\boldsymbol{\phi})$ is a function of the damage tensor $\boldsymbol{\phi}$ and is no longer a constant. Using equation (7.25) along with expressions (7.24) and (7.26), one obtains the following relation between $\bar{\mathbf{E}}$ and $\mathbf{E}(\boldsymbol{\phi})$:

$$E_{klmn}(\boldsymbol{\phi}) = M_{ijkl}^{-1}(\boldsymbol{\phi}) \bar{E}_{ijpq} M_{pqmn}^{-T}(\boldsymbol{\phi}) \quad (7.27)$$

where the superscript $-T$ indicates the transpose of the inverse of the tensor as depicted by $M_{ijkl}^{-1} M_{mnkl} = \delta_{im} \delta_{jn}$. Finally, using equation (7.23) along with equations (7.24), (7.25), and (7.27), one obtains the desired linear relation between the elastic strain tensor $\boldsymbol{\epsilon}'$ and its effective counterpart $\bar{\boldsymbol{\epsilon}}'$:

$$\bar{\epsilon}'_{kl} = M_{klmn}^{-T} \epsilon'_{mn} \quad (7.28)$$

The two transformation equations (7.27) and (7.28) will be incorporated later in this chapter in the general inelastic constitutive model that will be developed.

7.2.2 Effective Plastic Strain Rate

The constitutive model to be developed here is based on a von Mises type yield function $f(\boldsymbol{\sigma}', \mathbf{X}', \kappa, \boldsymbol{\phi})$ in the configuration C that involves both isotropic and kinematic hardening through the evolution of the plastic work κ and the backstress tensor $\mathbf{X}'$, respectively. The corresponding yield function $f(\bar{\boldsymbol{\sigma}}', \bar{\mathbf{X}}', \bar{\kappa}, \mathbf{0})$ in the configuration $\bar{C}$ is given by:

$$f = \frac{3}{2}(\bar{\sigma}'_{kl} - \bar{X}'_{kl})(\bar{\sigma}'_{kl} - \bar{X}'_{kl}) - \bar{\sigma}_o^2 - c\bar{\kappa} = 0 \quad (7.29)$$

where σ_o and c are material parameters denoting the uniaxial yield strength and isotropic hardening, respectively. The plastic work κ is a scalar function and its evolution in the configuration $\bar{C}$, is taken here to be in the following form:

$$d\bar{\kappa} = \sqrt{\bar{d}''_{ij}\bar{d}''_{ij}} \quad (7.30)$$

where $\bar{d}''_{ij}$ is the plastic part of the spatial strain rate tensor $\mathbf{d}$.

Isotropic hardening is described by the evolution of the plastic work κ as given above. In order to describe kinematic hardening, the Prager-Ziegler evolution law (Oldroyd, 1950) is used here in the configuration $\bar{C}$, as follows:

$$d\bar{\alpha}_{ij} = d\bar{\mu}(\bar{\sigma}'_{ij} - \bar{X}'_{ij}) \quad (7.31)$$

where $d\bar{\mu}$ is a scalar function to be determined shortly.

The plastic flow in the configuration $\bar{C}$ is described by the associated flow rule in the form:

$$\bar{d}''_{ij} = d\bar{\Lambda} \frac{\partial f}{\partial \bar{\sigma}_{ij}} \quad (7.32)$$

where $d\bar{\Lambda}$ is a scalar function introduced as a Lagrange multiplier in the constraint thermodynamic equations (see [section 7.3](#)), that is still to be determined. In the present formulation, it is assumed that the associated flow of plasticity will still hold in the configuration C , that is:

$$d''_{ij} = d\Lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (7.33)$$

where $d\Lambda$ is another scalar function that is to be determined.

Substituting the yield function f of equation (7.29) into equation (7.32) and using the transformation equations (7.5) and (7.15), one obtains:

$$\bar{d}''_{ij} = 3d\bar{\Lambda}N_{ijkl}(\sigma_{kl} - X_{kl}) \quad (7.34)$$

On the other hand, substituting the yield function f of equation (7.29) into equation (7.33) and noting the appropriate transformation (7.10) and (7.16), one obtains:

$$d''_{ij} = 3d\Lambda H_{ijkl}(\sigma_{kl} - X_{kl}) \quad (7.35)$$

It is noticed that plastic incompressibility exists in the configuration $\bar{C}$ as seen from equation (7.34) where $\bar{d}''_{mm} = 0$ since $N_{mmkl} = 0$. However,

this is not true in the configuration C since d''_{mm} does not vanish depending on $H_{m m k l}$ as shown in equation (7.35).

In order to derive the transformation equation between $\mathbf{d}''$ and $\bar{\mathbf{d}}''$, one first notices that:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \bar{\sigma}_{pq}} \frac{\partial \bar{\sigma}_{pq}}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \bar{\sigma}_{pr}} M_{pqij} \quad (7.36)$$

where M_{pqij} is defined in the equation (7.1) as $\partial \bar{\sigma}_{pq} / \partial \sigma_{ij}$. Using the above relation along with equations (7.32) and (7.33), one obtains:

$$\bar{d}''_{ij} = \frac{d\bar{\Lambda}}{d\Lambda} M_{ijmn}^{-1} d''_{mn} \quad (7.37)$$

The above equation represents the desired relation, except that the expression $d\bar{\Lambda}/d\Lambda$ needs to be determined. This is done by finding explicit expressions for both $d\Lambda$ and $d\bar{\Lambda}$ using the consisting conditions. The rest of this section is devoted to this task. But first one needs to determine an appropriate expression for $d\bar{\mu}$ that appears in equation (7.31), since it plays an essential role in the determination of $d\bar{\Lambda}$.

In order to determine an expression for $d\bar{\mu}$, one assumes that the projection of $d\bar{\mathbf{X}}'$ on the gradient of the yield surface f in the stress space is equal to $b\bar{\mathbf{d}}''$ in the configuration $\bar{C}$, where b is a material parameter to be determined from the uniaxial tension test (Voyiadjis, 1984; Voyiadjis and Kiousis, 1987). This assumption is written as follows:

$$b\bar{d}''_{kl} = d\bar{X}'_{mn} \frac{\frac{\partial f}{\partial \bar{\sigma}_{mn}}}{\frac{\partial f}{\partial \bar{\sigma}_{pq}} \frac{\partial f}{\partial \bar{\sigma}_{pq}}} \frac{\partial f}{\partial \bar{\sigma}_{kl}} \quad (7.38)$$

Substituting for $d\bar{\mathbf{X}}'$ and $\mathbf{d}''$ from equation (7.31) and (7.32), respectively, into equation (7.38) and post-multiplying the resulting equation by $\partial f / \partial \bar{\sigma}_{kl}$, one obtains the required expression for $d\bar{\mu}$:

$$d\bar{\mu} = b d\bar{\Lambda} \frac{\frac{\partial f}{\partial \bar{\sigma}_{mn}} \frac{\partial f}{\partial \bar{\sigma}_{mn}}}{(\bar{\sigma}'_{pq} - \bar{X}'_{pq}) \frac{\partial f}{\partial \bar{\sigma}_{pq}}} \quad (7.39)$$

Using the elastic linear relationship in equation (7.19), and taking its rate, one obtains:

$$d\bar{\sigma}_{ij} = \bar{E}_{ijkl} \bar{d}'_{kl} \quad (7.40)$$

where $\bar{d}'_{kl}$ is assumed to be equal to $d\bar{\epsilon}'_{kl}$ based on the assumption of small elastic strains as discussed in section 7.2.1. Eliminating $\bar{\mathbf{d}}'$ from equation (7.40) through the use of expressions (7.18) and (7.32), one obtains:

$$d\bar{\sigma}_{ij} = \bar{E}_{ijkl}(\bar{d}_{kl} - d\bar{\Lambda} \frac{\partial f}{\partial \bar{\sigma}_{kl}}) \quad (7.41)$$

The scalar multiplier $d\bar{\Lambda}$ is obtained from the consisting condition $df(\bar{\sigma}'_{kl}, \bar{X}'_{kl}, 0, \bar{\kappa}) = 0$ such that:

$$\frac{\partial f}{\partial \bar{\sigma}'_{kl}} d\bar{\sigma}'_{kl} + \frac{\partial f}{\partial \bar{X}'_{kl}} d\bar{X}'_{kl} + \frac{\partial f}{\partial \bar{\kappa}} d\bar{\kappa} = 0 \quad (7.42)$$

Using equations (7.18), (7.30), (7.32), (7.39), and (7.41) into equation (7.42), one obtains the following expression for $d\bar{\Lambda}$:

$$d\bar{\Lambda} = \frac{1}{\bar{Q}} \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} \bar{d}_{mn} \quad (7.43)$$

where $\bar{Q}$ is scalar and given by:

$$\bar{Q} = \frac{\partial f}{\partial \bar{\sigma}'_{ab}} \bar{E}_{abcd} \frac{\partial f}{\partial \bar{\sigma}'_{cd}} - \frac{\partial f}{\partial \bar{\kappa}} \sqrt{\frac{\partial f}{\partial \bar{\sigma}_{pq}} \frac{\partial f}{\partial \bar{\sigma}_{pq}}} - b \frac{\partial f}{\partial \bar{X}'_{ef}} (\bar{\sigma}'_{ef} - \bar{X}'_{ef}) \frac{\frac{\partial f}{\partial \bar{\sigma}_{ij}} \frac{\partial f}{\partial \bar{\sigma}_{ij}}}{(\bar{\sigma}'_{gh} - \bar{X}'_{gh}) \frac{\partial f}{\partial \bar{\sigma}_{gh}}} \quad (7.44)$$

Assuming that the Prager-Ziegler kinematic hardening rule holds in the configuration C along with the projection assumption of equation (7.13), one can derive a similar equation to (7.43), in the following form:

$$d\Lambda = \frac{1}{Q} \frac{\partial f}{\partial \sigma'_{kl}} E_{klmn} d_{mn} \quad (7.45)$$

where Q is given by:

$$Q = \frac{\partial f}{\partial \sigma'_{ab}} E_{abcd} \frac{\partial f}{\partial \sigma'_{cd}} - \frac{\partial f}{\partial \kappa} \sqrt{\frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{pq}}} - b \frac{\partial f}{\partial X'_{ef}} (\sigma'_{ef} - X'_{ef}) \frac{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}}{(\sigma'_{gh} - X'_{gh}) \frac{\partial f}{\partial \sigma_{gh}}} \quad (7.46)$$

In contrast to the method used by Voyiadjis and Kattan (1990) where the two yield functions in the configurations C and $\bar{C}$ are assumed to be equal, a more consistent approach is adopted here. This approach is based on the assumptions used to derive equation (7.45). It is clear that in this method, the two yield functions in the configurations C and $\bar{C}$ are treated separately and two separate consistency conditions are thus invoked. In the authors' opinion, this emphasizes a more consistent approach than the method used by Voyiadjis and Kattan (1990).

One is now left with the tedious algebraic manipulations of equations (7.43) and (7.45) in order to derive an appropriate form for the ratio $d\bar{\Lambda}/d\Lambda$.

First, equation (7.45) is rewritten in the following form, where the appropriate transformations $\bar{\mathbf{E}} \rightarrow \mathbf{E}$ and $\boldsymbol{\sigma} \rightarrow \bar{\boldsymbol{\sigma}}$ are used:

$$Q d\Lambda = \frac{\partial f}{\partial \bar{\sigma}'_{ij}} \bar{E}_{ijpq} M_{pqmn}^{-T} d_{mn} \quad (7.47)$$

Then, one expands equation (7.43) by using the appropriate transformations $\bar{\mathbf{d}}' \rightarrow \mathbf{d}'$ and $\bar{\mathbf{d}}'' \rightarrow \mathbf{d}''$ to obtain:

$$d\bar{\Lambda} = \frac{1}{Q} \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} [dM_{mnpq}^{-T} E_{ijpq}^{-1} \sigma_{ij} + M_{mnpq}^{-T} d_{pq} - (M_{mnpq}^{-T} - \frac{d\bar{\Lambda}}{d\Lambda} M_{mnpq}^{-1}) d''_{pq}] \quad (7.48)$$

where the derivative dM_{mnpq} is defined in the next section. The last major step in the derivation is to substitute the term on the right-hand side of equation (7.47) for the results using the transformations $\boldsymbol{\sigma} \rightarrow \bar{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma} \rightarrow \bar{\boldsymbol{\sigma}}'$, $\mathbf{E} \rightarrow \bar{\mathbf{E}}$, and others. Once this is done, the following relation is obtained:

$$\begin{aligned} (\bar{Q} - \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} \frac{\partial f}{\partial \bar{\sigma}_{mn}}) d\bar{\Lambda} &= (Q - \frac{\partial f}{\partial \sigma'_{ij}} E_{ijpq} \frac{\partial f}{\partial \sigma_{pq}}) d\Lambda \\ &+ \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} dM_{mnpq}^{-T} E_{ijpq}^{-1} \sigma_{ij} \end{aligned} \quad (7.49)$$

The above equation is rewritten in the form:

$$a_1 d\bar{\Lambda} = a_2 d\Lambda + a_3 \quad (7.50)$$

where

$$a_1 = \bar{Q} - \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} \frac{\partial f}{\partial \bar{\sigma}_{mn}} \quad (7.51a)$$

$$a_2 = Q - \frac{\partial f}{\partial \sigma'_{kl}} E_{klmn} \frac{\partial f}{\partial \sigma_{mn}} \quad (7.51b)$$

$$a_3 = \frac{\partial f}{\partial \bar{\sigma}'_{kl}} \bar{E}_{klmn} dM_{mnpq}^{-T} E_{ijpq}^{-1} \sigma_{ij} \quad (7.51c)$$

It is noticed that a_1 and a_2 are the last two terms on the right-hand sides of equations (7.44) and (7.46), respectively.

It should be noted that when the material undergoes only plastic deformation without damage, that is when the configurations C and $\bar{C}$ coincide, then $a_1 = a_2$ and $a_3 = 0$ since $d\mathbf{M}$ vanishes in this case, thus leading to $d\bar{\Lambda} = d\Lambda$.

The relation (7.50) is now substituted into equation (7.37) along with equation (7.35) to obtain the following nonlinear transformation equation for the plastic part of the spatial strain rate tensor:

$$\bar{d}_{ij}'' = X_{ijkl}d_{kl}'' + Z_{ij} \quad (7.52)$$

where the tensors $\mathbf{X}$ and $\mathbf{Z}$ are given by:

$$X_{ijkl} = \frac{a_2}{a_1} M_{ijkl}^{-1} \quad (7.53)$$

$$Z_{ij} = 3 \frac{a_3}{a_1} M_{ijkl}^{-1} H_{klmn} (\sigma_{mn} - X_{mn}) \quad (7.54)$$

The transformation equation (7.52) will be used later in the derivation of the constitutive equations.

7.3 Constitutive Model

In this section, a coupled constitutive model will be derived incorporating both elasto-plasticity and damage. This section is divided into three subsections detailing the derivation starting with the equations then proceeding to the desired coupling.

7.3.1 Damage Evolution

In this section, an inelastic constitutive model is derived in conjunction with the damage transformation equations presented in the previous sections. An elasto-plasticity stiffness tensor that involves damage effects is derived in the Eulerian reference system. In this formulation, rate-dependent effects are neglected and isothermal conditions are assumed. The damage evolution criterion to be used here is that proposed by Lee et al. (1985) and is given by:

$$g(\bar{\sigma}, L) = \frac{1}{2} J_{ijkl} \bar{\sigma}_{ij} \bar{\sigma}_{kl} - l_o^2 - L(l) \equiv 0 \quad (7.55)$$

where J_{ijkl} are the components of a constant fourth-order tensor that is symmetric and isotopic. This tensor is represented by the following matrix:

$$\mathbf{J} \equiv \begin{bmatrix} 1 & \mu & \mu & 0 & 0 & 0 \\ \mu & 1 & \mu & 0 & 0 & 0 \\ \mu & \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1-\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1-\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1-\mu) \end{bmatrix} \quad (7.56)$$

where μ is a material constant satisfying $-\frac{1}{2} \leq \mu \leq 1$. In equation (7.55), l_o represents the initial damage threshold, $L(l)$ is the increment of damage threshold, and l is the scalar variable that represents overall damage.

During the process of plastic deformation and damage, the power of dissipation Π is given by (Voyiadjis and Kattan, 1990):

$$\Pi = \sigma_{ij} d''_{ij} + \sigma_{kl} d\phi_{kl} - Ldl \quad (7.57)$$

In order to obtain the actual values of the parameters σ , ϕ , κ , and l , one needs to solve an extremization problem, i.e, the power of dissipation Π is to be extremized subject to two constraints, namely, $f(\sigma', \mathbf{X}', \kappa, \phi) = 0$ and $g(\bar{\sigma}, L) = 0$. Using the method of the calculus of functions of several variables, one introduces two Lagrange multipliers $d\lambda_1$, and $d\lambda_2$, and forms the function Ψ such that:

$$\Psi = \Pi - d\lambda_1 f - d\lambda_2 g \quad (7.58)$$

The problem now reduces to that of extremizing the function Ψ . Therefore, one uses the necessary conditions $\partial\Psi/\partial\sigma = 0$ and $\partial\Psi/\partial L = 0$ to obtain:

$$d''_{ij} + d\phi_{ij} - d\lambda_1 \frac{\partial f}{\partial \sigma_{ij}} - d\lambda_2 \frac{\partial g}{\partial \sigma_{ij}} = 0 \quad (7.59a)$$

$$-dl - d\lambda_2 \frac{\partial g}{\partial L} = 0 \quad (7.59b)$$

Next, one obtains from equation (7.55) that $\partial g/\partial L = -1$. Substituting this into equation (7.59b), one obtains $d\lambda_2 = dl$. Thus $d\lambda_2$ describes the evolution of the overall damage parameter l which is to be determined shortly. Using equation (7.59b) and assuming that damage and plastic deformation are two independent processes, one obtains the following two rate equations for the plastic strain and damage tensors:

$$d''_{ij} = d\lambda_1 \frac{\partial f}{\partial \sigma_{ij}} \quad (7.60a)$$

$$d\phi_{ij} = dl \frac{\partial g}{\partial \sigma_{ij}} \quad (7.60b)$$

The first equation of (7.60) is the associated flow rule for the plastic strain introduced earlier in equation (7.33), while the second is the evolution of the damage tensor. It is to be noted that $d\lambda_1$ is exactly the same as the multiplier $d\Lambda$ used earlier. However, one needs to obtain explicit expressions for the multipliers $d\Lambda$ and dl . The derivation of an expression for $d\Lambda$ will be left for the next section when the inelastic constitutive model is discussed. Now one proceeds to derive an expression for dl . This is done by invoking the consistency condition $dg(\boldsymbol{\sigma}, \boldsymbol{\phi}, L) = 0$. Therefore, one obtains:

$$\frac{\partial g}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial g}{\partial \phi_{kl}} d\phi_{kl} + \frac{\partial g}{\partial L} dL = 0 \quad (7.61)$$

Substituting for $d\phi$ from equation (7.60b) along with $\partial g / \partial L = -1$ and $dL = dl(\partial L / \partial l)$, we obtain:

$$dl = \frac{\frac{\partial g}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial L}{\partial l} - \frac{\partial g}{\partial \phi_{pq}} \frac{\partial g}{\partial \sigma_{pq}}} \quad (7.62)$$

Finally, by substituting equation (7.62) into equation (7.60b), we obtain the general evolution equation for the damage tensor $\boldsymbol{\phi}$ as follows:

$$d\phi_{kl} = \frac{\frac{\partial g}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial L}{\partial l} - \frac{\partial g}{\partial \phi_{pq}} \frac{\partial g}{\partial \sigma_{pq}}} \frac{\partial g}{\partial \sigma_{kl}} \quad (7.63)$$

The evolution equation (7.63) is to be incorporated in the constitutive model in the next two sections. It will also be used in the derivation of the elasto-plastic stiffness tensor. It should be noted that equation (7.63) is based on the damage criterion of equation (7.55) which is applicable to anisotropic damage. However, using the form for $\mathbf{J}$ given in equation (7.56) restricts the formulation to isotropy.

7.3.2 Plastic Deformation

In the analysis of finite strain plasticity, one needs to define an appropriate corotational stress rate that is objective and frame-indifferent. Detailed discussions of the types of stress rates are available in the papers of Voyiadjis and Kattan (1989) and Paulun and Pecherski (1985). The corotational stress rate to be adopted in this model is given for $\boldsymbol{\sigma}$ in the following form:

$$\overset{\circ}{\sigma}_{ij} = d\sigma_{ij} - \Omega_{ip}\sigma_{pj} + \sigma_{iq}\Omega_{qj} \quad (7.64)$$

where the modified spin tensor $\mathbf{\Omega}$ is given by (Paulun and Pecherski, 1985; Voyiadjis and Kattan, 1989) as follows:

$$\Omega_{ij} = \omega W_{ij} \quad (7.65)$$

In equation (7.65), $\mathbf{W}$ is the material spin tensor (the antisymmetric part of the velocity gradient) and ω is an influence scalar function too be determined. The effect of ω on the evolution of the stress and backstress is discussed in detail by Voyiadjis and Kattan (1989). The corotational rate $\overset{\circ}{\mathbf{\alpha}}$ has a similar expression as that in equation (7.64) keeping in mind that the modified spin tensor $\mathbf{\Omega}$ remains the same in both equations.

The yield function to be used in this model is the function f given by equation (7.29) with both isotropic and kinematic hardening. Isotropic hardening is described by the evolution of the plastic work as given earlier by equation (7.30), while kinematic hardening is given by equation (7.31). Most of the necessary plasticity equations were given in [section 7.2.2](#) and the only thing remaining is the derivation of the constitutive equation.

By substituting for $d\bar{\Lambda}$ from equation (7.43) into equation (7.41), one derives the general inelastic constitutive equation in the configuration $\bar{C}$ as follows:

$$d\bar{\sigma}_{ij} = \bar{D}_{ijkl} d\bar{e}_{kl} \quad (7.66)$$

where the elasto-plastic stiffness tensor $\mathbf{D}$ is given by:

$$\bar{D}_{ijkl} = \bar{E}_{ijkl} - \frac{1}{\bar{Q}} \frac{\partial f}{\partial \bar{\sigma}'_{mn}} \bar{E}_{mnkl} \bar{E}_{ijpq} \frac{\partial f}{\partial \bar{\sigma}_{pq}} \quad (7.67)$$

The next step is to use the transformation equations developed in the previous sections in order to obtain a constitutive equation in the configuration C similar to equation (7.66).

7.3.3 Coupling of Damage and Plastic Deformation

In this section, the transformation equations developed in [sections 7.1](#) and [7.2](#) are used with the constitutive model of the previous section in order to transform the inelastic constitutive equation (7.66) in the configuration $\bar{C}$ to a general constitutive equation in the configuration C that accounts for both damage and plastic deformation.

Using equation (7.28) and taking its derivative, one obtains the following transformation equation for $\bar{\mathbf{d}}'$:

$$\bar{d}'_{ij} = dM_{ijmn}^{-T} \epsilon'_{mn} + M_{ijkl}^{-T} d'_{kl} \quad (7.68)$$

where $d\mathbf{M}^{-T}$ is obtained by taking the derivative of the identity $\mathbf{M}^T\mathbf{M}^{-T} = \mathbf{I}$ and noting that $d\mathbf{I} = \mathbf{0}$. Thus, one has:

$$dM_{klmn}^{-T} = -M_{ijkl}^{-T}dM_{ijpq}^T M_{pqmn}^{-T} \quad (7.69)$$

The derivative $d\mathbf{M}$ is obtained by using the chain rule as follows:

$$dM_{ijpq} = \frac{\partial M_{ijpq}}{\partial \phi_{mn}} d\phi_{mn} \quad (7.70a)$$

Also, the corotational derivative $\overset{\circ}{\mathbf{M}}$ may be used defined by the following Lie derivative given by Oldroyd (1950):

$$\overset{\circ}{M}_{ijmn} = dM_{ijmn} - \Omega_{ip}M_{pjm n} - \Omega_{jq}M_{iqm n} - \Omega_{mr}M_{ijr n} - \Omega_{ns}M_{ijm s} \quad (7.70b)$$

The transformation equation (7.68) for the effective elastic strain rate tensor $\bar{\mathbf{d}}'$ represents a nonlinear relation unlike that of the effective stress tensor of equation (7.1). A similar nonlinear transformation equation (7.52) was previously derived for the effective plastic strain $\mathbf{d}''$. These two equations will now be used in the derivation of the constitutive model.

Now we are ready to derive the inelastic constitutive relation in the configuration C . Starting with the constitutive equation (7.66) and substituting for $\bar{\sigma}_{ij}$ and $\bar{d}_{kl}$ from equations (7.1) and (7.18), respectively, along with equations (7.68) and (7.52), we obtain:

$$dM_{ijmn}\sigma_{mn} + M_{ijpq}d\sigma_{pq} = \bar{D}_{ijkl}(M_{klmn}^{-T}d'_{mn} + dM_{klpq}^{-T}\epsilon'_{pq} + X_{klmn}d''_{mn} + Z_{kl}) \quad (7.71)$$

Next, we substitute for $d\mathbf{M}$ and $d\mathbf{M}^{-T}$ from equations (7.70) and (7.69), respectively, for $\mathbf{d}''$ from equation (7.18), and for $\mathbf{d}'$ from a similar equation to (7.40), i.e., $d'_{ij} = E_{ijkl}^{-1}d\sigma_{kl}$, into equation (7.71), the resulting expression is:

$$\begin{aligned} \frac{\partial M_{ijmn}}{\partial \phi_{pq}} d\phi_{pq} \sigma_{mn} + M_{ijpq} d\sigma_{pq} &= \bar{D}_{ijkl} (M_{klmn}^{-T} E_{pqmn}^{-1} d\sigma_{pq} - \\ M_{xykl}^{-T} \frac{\partial M_{xyuv}^T}{\partial \phi_{mn}} d\phi_{mn} M_{uvpq} E_{cdpq}^{-1} \sigma_{cd} &+ X_{klmn} d_{mn} - X_{klmn} E_{abmn}^{-1} d\sigma_{ab} + Z_{kl}) \end{aligned} \quad (7.72)$$

Finally, we substitute for $d\phi$ from equation (7.63) into equation (7.72) and solve for $d\sigma$ in terms of $\mathbf{d}$. After several algebraic manipulations, we obtain the desired inelastic constitutive relation in the configuration C as follows:

$$d\sigma_{ij} = D_{ijkl}d_{kl} + G_{ij} \quad (7.73)$$

where the effective elasto-plastic stiffness tensor $\mathbf{D}$ and the additional tensor $\mathbf{G}$ (comparable to the plastic relaxation stress introduced by Simo and Ju (1987)) are given by:

$$D_{ijkl} = O_{pqij}^{-1} \bar{D}_{pqmn} X_{mnkl} \quad (7.74a)$$

$$G_{ij} = O_{pqij}^{-1} \bar{D}_{pqmn} Z_{mn} \quad (7.74b)$$

where the fourth-order tensor $\mathbf{O}$ is given by:

$$\begin{aligned} O_{ijpq} = M_{ijpq} + \frac{\frac{\partial M_{ijmn}}{\partial \phi_{uv}} \frac{\partial q}{\partial \sigma_{pq}} \frac{\partial q}{\partial \sigma_{uv}} \sigma_{mn}}{\frac{\partial L}{\partial l} - \frac{\partial q}{\partial \phi_{xy}} \frac{\partial q}{\partial \sigma_{xy}}} - \bar{D}_{ijkl} (M_{klmn}^{-T} E_{pqmn}^{-1} - X_{klmn} E_{pqmn}^{-1} \\ - M_{xykl}^{-T} \frac{\partial M_{xyuv}^{-T}}{\partial \phi_{mn}} \frac{\frac{\partial q}{\partial \sigma_{pq}} \frac{\partial q}{\partial \sigma_{mn}}}{\frac{\partial L}{\partial l} - \frac{\partial q}{\partial \phi_{ab}} \frac{\partial q}{\partial \sigma_{ab}}} M_{uvcd}^{-T} M_{rscd}^{-T} \sigma_{rs}) \end{aligned} \quad (7.75)$$

The effective elasto-plastic stiffness tensor $\mathbf{D}$ in equation (7.74a) is the stiffness tensor including the effects of damage and plastic deformation. It is derived in the configuration of the deformed and damaged body. Equations (7.74) can now be used in finite element analysis. However, it should be noted that the constitutive relation in equation (7.73) represents a nonlinear transformation that makes the numerical implementation of this model impractical. This is due to the additional term G_{ij} which can be considered as some residual stress due to the damaging process. Nevertheless, the constitutive equation becomes linear provided that $G_{ij} = 0$. This is possible only when the term $(\sigma_{mn} - X_{mn})N_{ijrr}$ vanishes as seen in equations (7.74b) and (7.54) and therefore:

$$d\sigma_{ij} = D_{ijkl}d_{kl} \quad (7.76)$$

Upon investigation of the nonlinear constitutive equation (7.73), it is seen that the extra term G_{ij} is due to the linear transformation of the effective stress σ and $\bar{\sigma}$ in equation (7.1). It was shown in equation (7.4) that this transformation leads to a nonlinear relation between σ' and $\bar{\sigma}'$. The authors have shown in a recent work (Voyiadjis and Kattan, 1999) that a linear constitutive equation similar to equation (7.76) can be obtained if a linear transformation is assumed between the deviatoric stresses σ' and $\bar{\sigma}'$ in the form $\bar{\sigma}'_{ij} = M_{ijkl}\sigma'_{kl}$.

For completeness, one can obtain an identity that may be helpful in the numerical calculations. This is done by using the plastic volumetric incompressibility condition (which results directly from equation (7.34)):

$$\bar{d}''_{kk} = 0 \quad (7.77)$$

in the configuration $\bar{C}$. Equation (7.77) is commonly used in metal plasticity without damage (Paulun and Pechervski, 1985; Voyiadjis and Kattan, 1989). Using equation (7.34) along with the condition (7.77), one obtains the useful identity:

$$N_{rrkl}(\sigma_{kl}^{\circ} - X_{kl}) = 0 \quad (7.78)$$

Equation (7.78) is consistent with the previous conclusion of equation (7.7) since it was shown earlier that $N_{rrkl} = 0$.

In finite element calculations the critical state of damage is reached when the overall parameter l reaches a critical value called l_{cr} in at least one of the element. This value determines the initiation of microcracks and other damaging defects. Alternatively, one can assign several critical values $l_{cr}^{(1)}$, $l_{cr}^{(2)}$, etc. for different damage effects. In order to determine these critical values, which may be considered as material parameters, a series of uniaxial extension tests are to be performed on tensile specimens and the stress-strain curves drawn.

In order to determine $l_{cr}^{(i)}$ (the value of l at which damage initiation starts for a particular damage process "i"), the tensile specimen has to be sectioned at each load increment. The cross-section is to be examined for any cracks or cavities. The load step when cracks first appear in terms of the strain ϵ_1 is to be recorded and compared with the graph of l vs ϵ_1 . The corresponding value of l obtained in this way will be taken to be the critical value for l_{cr} . This value is to be used in the finite element analysis of more complicated problems. For more details, see the papers by Chow and Wang (1988) and Voyiadjis (1984, 1988).

7.4 Application to Void Growth: Gurson's Model

Gurson (1977) proposed a yield function $f(\sigma, v)$ for a porous solid with a randomly distributed volume fraction of voids. Gurson's model was used later (Tvergaard, 1982; Tvergaard and Needleman, 1984) to study necking and failure of damaged solids. Tvergaard and Needleman (1984) modified Gurson's yield function in order to account for rate sensitivity and necking

instabilities in plastically deforming solids. The modified yield function is used here in the following form (which includes kinematic hardening):

$$f = (\sigma'_{ij} - X'_{ij})(\sigma'_{ij} - X'_{ij}) + 2q_1\sigma_F^2 v \cosh\left(\frac{\sigma_{kk}}{2\sigma_F}\right) - \sigma_F^2(1 + q_2v^2) = 0 \quad (7.79)$$

where σ_F is the yield strength of the matrix material and q_1 and q_2 are material parameters introduced by Tvergaard (1982) to improve agreement between Gurson's model and other results. In equation (7.79), the variable v denotes the void volume fraction in the damaged material. In Gurson's model, damage is characterized by void growth only. The void growth is described by the rate of change of v given by (Voyiadjis, 1988):

$$dv = (1 - v)d''_{kk} \quad (7.80)$$

In Gurson's model, it is assumed that the voids remain spherical in shape through the whole process of deformation and damage. The change of shape of voids, their coalescence and nucleation of new voids are ignored in the model. Equation (7.80) implies also that the plastic volumetric change, d''_{kk} , does not vanish for a material with voids.

In the following, it is shown how the proposed model outlined in the first sections of this chapter can be used to obtain the damage effect tensor $\mathbf{M}$ as applied to Gurson's yield function. It is also shown how certain expressions can be derived for the parameters q_1 and q_2 in a consistent manner. One first starts with the yield function f in the configuration C . Therefore, using equation (7.29) in the form:

$$f = (\bar{\sigma}'_{mn} - \bar{X}'_{mn})(\bar{\sigma}'_{mn} - \bar{X}'_{mn}) - \frac{2}{3}\bar{\sigma}_o^2 \quad (7.81)$$

where the term $c\bar{\kappa}$ is dropped since isotropic hardening is not displayed by Gurson's function. Using the transformation equations (7.10), (7.16), and (7.17) and noting that $\sigma_F^2 = 2\bar{\sigma}_o^2/3$, equation (7.81) becomes:

$$f = H_{ijkl}(\sigma_{ij} - X_{ij})(\sigma_{kl} - X_{kl}) - \sigma_F^2 \quad (7.82)$$

It is noticed that equation (7.81) corresponds exactly to Gurson's function of equation (7.79) with $v = 0$. Using equation (7.2) and (7.12) to transform the total stresses in equation (7.82) into deviatoric stresses, one obtains:

$$f = H_{ijkl}(\sigma'_{ij} - X'_{ij})(\sigma'_{kl} - X'_{kl}) + \frac{1}{9}H_{mmnn}(\sigma_{pp} - X_{qq})^2 - \sigma_F^2 \quad (7.83)$$

Equation (7.83) represents the yield function f in the configuration C , which can now be compared to Gurson's yield function of equation (7.79).

Thus, upon comparing equation (7.79) with equation (7.83), it is clear that the deviatoric parts of the two functions must be equal. Therefore, one obtains:

$$H_{ijkl}(\sigma'_{ij} - X'_{ij})(\sigma'_{kl} - X'_{kl}) = (\sigma'_{rs} - X'_{rs})(\sigma'_{rs} - X'_{rs}) \quad (7.84)$$

On the other hand, upon equating the remaining parts of the two functions, one obtains:

$$\frac{1}{9}H_{mmnn}(\sigma_{pp} - X_{qq})^2 = 2q_1\sigma_F^2v \cosh\left(\frac{\sigma_{kk}}{2\sigma_F}\right) - q_2\sigma_F^2v^2 \quad (7.85)$$

The problem is now reduced to manipulating equations (7.84) and (7.85). Rewriting equation (7.84) in the following form:

$$(H_{ijkl} - \delta_{ik}\delta_{jl})(\sigma'_{ij} - X'_{ij})(\sigma'_{kl} - X'_{kl}) = 0 \quad (7.86)$$

One concludes that the tensor $\mathbf{H}$ is constant for Gurson's model and can be expressed as follows:

$$H_{ijkl} = \delta_{ik}\delta_{jl} \quad (7.87)$$

It is clear that the deviatoric part of Gurson's yield function does not display any damage characteristics as given by equation (7.79). This is further supported by equation (7.87) where the damage effect tensor is independent of the damage variable ϕ . Next, upon considering equation (7.87), one obtains $H_{mmnn} = 3$. Substituting this value into equation (7.85) yields:

$$\frac{1}{3}(\sigma_{pp} - X_{qq})^2 = [2q_1\cosh\left(\frac{\sigma_{kk}}{2\sigma_F}\right) - q_2v]v\sigma_F^2 \quad (7.88)$$

Equation (7.88) must be satisfied for a possible relationship between Gurson's model and the proposal model. Equation (7.88), as it stands, does not seem to merit an explicit relationship between parameters q_1 , q_2 , and v . This is due to the presence of the "cosh" term on the right-hand side. Therefore, it is clear that one cannot proceed further without making some assumptions. In particular, two assumptions are to be employed. The first assumption is valid for small values of $\sigma_{kk}/(2\sigma_F)$, where only the first two terms in the "cosh" series expansion are considered:

$$\cosh\left(\frac{\sigma_{kk}}{2\sigma_F}\right) = 1 + \frac{\sigma_{kk}^2}{8\sigma_F^2} \quad (7.89)$$

The second assumption concerns the term X_{qq} which appears in the equation (7.88). For the following to be valid, one needs to consider a modified Gurson yield function where the volumetric stress σ_{kk} is replaced

by $(\sigma_{kk} - X_{qq})$. Therefore, upon incorporating the above two assumptions into equation (7.88), one obtains:

$$\frac{1}{3}(\sigma_{kk} - X_{qq})^2 = \frac{1}{4}q_1 v (\sigma_{kk} - X_{qq})^2 + (2q_1 - q_2 v) v \sigma_F^2 \quad (7.90)$$

It is now clear from the equation (7.90) that the following two expressions for q_1 and q_2 in terms of v , need to be satisfied:

$$q_1 = \frac{4}{3v} \quad (7.91a)$$

$$q_2 = \frac{8}{3v^2} \quad (7.91b)$$

The relations (7.91a,b) represent variable expressions for the parameters q_1 and q_2 in terms of the void volume fraction, in contrast to the constant values that were suggested earlier by various authors. The relations (7.91a,b) are consistently derived and although they are approximate, in the authors' opinion, they form a basis for more sophisticated expressions. In addition, they are based on a solid derivation which cannot be said for the constant values that were used in the literature. Finally, one more important point that came up in the derivation needs to be considered. As it stands, Gurson's function of equation (7.79) cannot be related to the work presented here. It is a modified version of Gurson's function containing the term $\cosh[(\sigma_{kk} - X_{qq})/(2\sigma_F)]$ instead of $\cosh[\sigma_{kk}/(2\sigma_F)]$ that is used in the derivation of equations (7.91a,b). The authors believe that this point should be pursued and the proposed modified Gurson function explored further.

7.5 Effective Spin Tensor

In this section, a formal derivation is presented for the transformation equation of the modified spin tensor that is used in the corotational rate equations. In the configuration $\bar{C}$, the corotational derivative of the effective Cauchy stress tensor is given by:

$$\overset{\circ}{\bar{\sigma}}_{ij} = d\bar{\sigma}_{ij} - \bar{\Omega}_{ip}\bar{\sigma}_{pj} + \bar{\sigma}_{iq}\bar{\Omega}_{qj} \quad (7.92)$$

where $\bar{\Omega}$ is the effective modified spin tensor. The problem now reduces to finding a relation between Ω and $\bar{\Omega}$. One should keep in mind, however, that equation (7.92) is valid only when a Cartesian coordinate system is used. The same remark applies to equation (7.64) in the configuration C .

In order to derive the required relation, one first starts with the transformation equation (7.1). Taking the corotational derivative of this equation and rearranging the terms, one obtains:

$$\overset{\circ}{\sigma}_{ke} = M_{ijke}^{-1} (\overset{\circ}{\sigma}_{ij} - \overset{\circ}{M}_{ijmn} \sigma_{mn}) \quad (7.93)$$

Substituting for $\overset{\circ}{\sigma}_{ij}$ from equation (7.92) into equation (7.93) and using the material time derivative $d\sigma_{ij} = dM_{ijrs}\sigma_{rs} + M_{ijpq}d\sigma_{pq}$, one obtains:

$$\begin{aligned} \overset{\circ}{\sigma}_{kl} = & M_{ijkl}^{-1} dM_{ijrs}\sigma_{rs} + d\sigma_{kl} - M_{ijkl}^{-1} \bar{\Omega}_{ip} M_{pjab} \sigma_{ab} + M_{ijkl}^{-1} M_{iqcd} \sigma_{cd} \bar{\Omega}_{qj} \\ & - M_{ijkl}^{-1} M_{ijmn} \sigma_{mn} \end{aligned} \quad (7.94)$$

Comparing the two corotational derivatives appearing in equations (7.64) and (7.94), and after performing some tedious algebraic manipulations, one can finally obtain a relation between $\mathbf{\Omega}$ and $\bar{\mathbf{\Omega}}$ in the following form:

$$\bar{\Omega}_{mn} = A_{mnpq} \Omega_{pq} + B_{mn} \quad (7.95)$$

where the tensors $\mathbf{A}$ and $\mathbf{B}$ are given by:

$$A_{mnpq} = -C_{mnkl}^{-1} (\delta_{kp} \sigma_{ql} - \delta_{lq} \sigma_{kp}) \quad (7.96a)$$

$$B_{mn} = -C_{mnkl}^{-1} M_{pqkl}^{-1} (dM_{pqrs} - \overset{\circ}{M}_{pqrs}) \sigma_{rs} \quad (7.96b)$$

and the tensor $\mathbf{C}$ is given by:

$$C_{klip} = (M_{qpkl}^{-1} M_{qicd} - M_{ijkl}^{-1} M_{pjcd}) \sigma_{cd} \quad (7.96c)$$

With the availability of the transformation equation of the spin tensor, the theory presented in this chapter is now complete.

Problems

- 7.1. Explain why equation (7.4) does not represent a linear transformation?
- 7.2. Derive equation (7.6) explicitly.
- 7.3. Derive equations (7.8) and (7.9) explicitly.
- 7.4. Derive equation (7.11) explicitly.
- 7.5. Derive equations (7.27) and (7.28) explicitly.

- 7.6. Assume a linear transformation in the form $\bar{\sigma}'_{ij} = M_{ijkl}\sigma'_{kl}$. Derive the corresponding elasto-plastic constitutive model for this case. Specifically, derive a new elasto-plastic stiffness tensor D_{ijkl} .
- 7.7. Derive equations (7.34) and (7.35) explicitly.
- 7.8. Derive equation (7.43) explicitly.
- 7.9. Derive equation (7.45) explicitly.
- 7.10. Derive equations (7.50) and (7.52) explicitly.
- 7.11. Consider a von Mises type yield function of the form given by equation (7.29). Neglect isotropic hardening and derive a new elasto-plastic constitutive model based on the modified yield criterion.
- 7.12. Consider a von Mises type yield function of the form given by equation (7.29). Neglect kinematic hardening and derive a new elasto-plastic constitutive model based on the modified yield criterion.
- 7.13. Consider a von Mises type yield function of the form given by equation (7.29). Neglect both isotropic and kinematic hardening, and derive a new elasto-plastic constitutive model based on the modified yield criterion.
- 7.14. Explore using a Tresca type yield function instead of a von Mises in the formulation of the elasto-plastic constitutive model given in this chapter.
- 7.15. Consider using a different damage evolution criterion by introducing the generalized force Y_{ij} associated with ϕ_{ij} in the equation (7.55). Derive a new evolution equation for the damage variable ϕ in this case.
- 7.16. Derive equations (7.62) and (7.63) explicitly.
- 7.17. Derive equation (7.67) explicitly.
- 7.18. Derive equation (7.70b) explicitly.
- 7.19. Derive equations (7.73), (7.74) and (7.75) explicitly.
- 7.20. Explain why there is an additional term G_{ij} in the elasto-plastic constitutive equation (7.73). What are the conditions under which this term vanishes.
- 7.21. Derive equations (7.87) and (7.88) explicitly.
- 7.22. Derive equation (7.89) explicitly.

- 7.23. Derive equation (7.91a,b) explicitly.
- 7.24. Derive equations (7.95) and (7.96) explicitly.