



Three-dimensional nonlinear analysis of adhesively bonded lap joints considering viscoplasticity in adhesives

P.C. Pandey^{*}, S. Narasimhan¹

Department of Civil Engineering, Indian Institute of Science, Bangalore 560 012, India

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Abstract

The paper presents three-dimensional (3D) viscoplastic analysis of adhesively bonded single lap joint considering material and geometric nonlinearity. The constitutive relations for the adhesive is developed using a pressure dependent (modified) von Mises yield function and Ramberg–Osgood idealization for the experimental stress–strain curve. The adherends and adhesive layers are modelled using 20 noded solid elements. The overall geometry of the lap joint, the loading and the boundary conditions have been considered both according to the ASTM testing standard as well as from those adopted in earlier investigations. Materially and geometrically nonlinear finite element analysis have been carried out on several joint configurations. However, observations have been made in particular on peel and shear stresses in the adhesive layer, which provides useful insight into the 3D nature of the problem. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Adhesive bonding technology has expanded greatly in recent years as more and more advanced composite materials are being utilized. Research and development of this technology covers over the past 50 years and has been mainly directed towards the requirements of the aerospace industry. Today, however, modern adhesives offer a joining technique of interest to engineers in a far wider range of industries. Examples of such applications are the sonar transducer adhesively bonded acoustical window and the likely necessity of composite structural components of carrier based aircraft. The civil engineering sector is becoming increasingly aware of adhe-

sives as a method of joining. However, their use to date has largely been restricted to essentially light bonding applications. The use of adhesive bonding as a joining method in aircraft construction is an accepted means of attaining high structural efficiency and improved fatigue life.

Also the adhesive bonding gives the light, stiff and economical structures free of blemishes caused by conventional assembly methods. It is observed from the various experiments and analysis that adhesive joints prove to be more efficient for lightly loaded structures, whereas mechanically fastened joints are more efficient for heavily loaded structures. Adhesive bonded joints have the major advantages of having less sources of stress concentration, efficient load transfer in large area of bonding, superior fatigue resistance and high strength–weight ratio compared to discrete joints.

Adhesives being viscous, flow over the surface of solid and because of their intimate contact, interact with its molecular forces. Then, as a result of adhesive curing process, they become strong solids which while retaining intimate contact with the surfaces hold them together.

^{*}Corresponding author. Tel.: +91-80-309-2667; fax: +91-80-334-1683.

¹Now at: Department of Ship Science, University of Southampton, Southampton, UK.

E-mail addresses: pcpandey@civil.iisc.ernet.in (P.C. Pandey), narasim@ship.soton.ac.uk (S. Narasimhan).

These adhesives are not as strong as metal adherends, and hence, the adhesive interlayer will always tend to be the weakest link in bonded structures. Care is therefore to be taken to ensure that service stresses are well within its capabilities. This is normally achieved by providing a relatively large area in bonding. The failure in adhesively bonded joints can occur due to any of the following reasons: cohesive failure within adhesive, adhesive failure which occurs at interface of adhesive and adherend and failure of adherends which also include delamination in composite structures or due to their combinations.

The other type of failure is cyclic debonding in which progressive separations of adherends occur by failure of adhesive under cyclic loading. The main cause of failure in adhesively bonded joints are the brittle nature of adhesives. The adhesive is more prone to damage when the structure is shock loaded and a momentary distortion of substrate generates powerful peel and cleavage forces that they may not be able to resist. Hence, an accurate analysis of bonded joints are needed in order to determine the stress distribution within adhesive for worst combination of load which is going to come on the structure.

The main problem involved in the analysis of adhesive bonding of structural systems is the structural complexities and modelling of viscous properties of adhesives.

As it has been pointed out that the stress pattern around the joint has a high stress gradient, idealization of the structure as plane stress/strain or as bending element requires high engineering judgement. Even the simplest of the joints, such as single lap joint under tensile forces, undergoes combined action of bending and stretching and leads to complications in the stress flow in three-dimensional (3D) pattern. Thus, in the finite element model, structural model of 3D behaviour.

Adhesives are viscous due to their material properties, which have hydrostatic (mean) stress dependent yield criterion, different yield strengths in tension and compression, nonlinear time dependent stress–strain relationship and sensitivity to temperature and humidity.

The nonlinear time dependent stress–strain relationships cannot be adequately represented by the viscoelastic theory even at moderate stress levels. The viscoelastic modelling only predicts the time dependent behaviour in elastic domains. However, adhesives show more time dependence at high stress levels, i.e., in plastic regions (viscoplastic).

2. Literature review

A review of the literature shows that most studies on single lap joint model the structure as two-dimensional (2D) component. All of them have considered the adhesively bonded joint either as a plane stress or a plane

strain case. A comprehensive literature review of both the analytic and the finite element technique for 2D analysis can be referred in Refs. [11,12]. The authors have performed parametric 2D viscoplastic analyses of adhesive lap joints considering geometric nonlinearity. Carpenter [2] has pioneered the finite element analysis for adhesive joints with the acceptable idealization for adhesive layers and has compared the finite element results with the analytical solutions. 3D finite element viscoplastic analysis with geometric nonlinearity of lap joints is studied by Narasimhan [7] and Pandey et al. [9,10]. Tsai and Morton [21] analysed the 3D nature of a single lap joint specimen in a linear elastic finite element analysis in which boundary conditions accounts for the geometrically nonlinear effects. It has been shown that a 3D region exists in the joint specimen, where adherend and adhesive stress distributions in the overlap length near (and especially on) the free surface are quite different from those occurring in the interior. It is further observed that the adhesive peel stress is extremely sensitive to 3D effect. The maximum value of peel stress occurs at the end of the overlap in the central 2D core region, rather at the corners where the 3D effects are found. Pickett and Hollaway [14,15] have presented the finite element analysis of single, double and tubular lap joints. They have considered a linearly elastic behaviour for both the adhesive and the adherends. The single lap joint has been analysed by considering the geometric nonlinearity, where as the double and the tubular lap joints were analysed without considering the geometric nonlinearity. They have used the program developed by Zienkiewicz (a total Lagrangian method) [23]. The linear finite element formulation which does not account for the joint rotation during the load, is shown to overestimate the peak adhesive stresses, where as a close agreement is reached for the peak values in the classical and geometric nonlinear finite element analysis of a single lap joint. However, for a double lap joint, the classical and the finite element solutions differ by 25% in their prediction of the peak stress. They concluded that this is due to the assumption that the boundary stress resultants are simple tension forces only and any bending effects which result from the eccentric loading will be ignored in the classical theory. In the tubular lap joint, maximum peel stresses differ by 40% with the finite element results being consistently higher than the classical results. They attributed this due to the error in the boundary conditions.

Harris and Adams [5] have presented the strength prediction of a single lap joint. They modelled the adherend and the adhesive as elastoplastic. They have analysed the single lap joint with edge spew fillet and considering the geometric nonlinearity, but the type of geometric nonlinear formulation used is not given. They have found that the maximum stress decreases with the increase in the applied load, when the geometric non-

linearity is considered and increased in the strength of the joint. The finite element solutions were compared with the closed form solutions and discrepancies were found in predicting the maximum stresses.

Roy and Reddy [16] have given the 2D finite element based on the updated Lagrangian formulation of elastic solids. The procedure is the same as that given by Bathe et al. [1]. Who have analysed the single lap joints considering the geometric nonlinearity and compared the results obtained from another program, VISTA. The results were in good agreement with the analysis. They have also studied the nonlinear response of a bonded cantilever plate under distributed transverse loading with two different meshes. One with a fine mesh and the other with a coarse mesh. There were discrepancies in the results obtained using both the meshes. They have studied the effect of boundary conditions on the adhesive stress distribution in a single lap joint.

Roy and Reddy [17,18] have analysed the adhesive bonded joints considering the geometric nonlinearity and modelled the adhesive as viscoelastic, but the adherends as linearly elastic. Large displacements and rotations, but small strains were accounted for the updated Lagrangian description of the motion. Dattaguru et al. [3] have studied the failure of a single lap joint by considering the geometric nonlinearity. They have concluded that nonlinear effects due to large rotations have significant effects on the strain energy release rates.

A unified finite element formulation for the geometrically nonlinear analysis is presented in Refs. [7,10]. Geometrically nonlinear analysis can be formulated in either a total Lagrangian or Eulerian coordinate system, first one in terms of the initial position and second one in terms of the final deformed state. From the computational point of view, Eulerian formulation is strictly an upgraded Lagrangian approach where the initial position becomes the current equilibrium state prior to some incremental stage. Stricklin et al. [20], Bathe et al. [1], Schrefler et al. [19] and Wood et al. [22] discussed the updated Lagrangian and the total Lagrangian formulations and pointed out that the total Lagrangian approach offers advantages since the initial configuration remains constant which simplifies the formulation and the computation. An advantage of the total Lagrangian formulation is that the derivatives of the interpolation functions are with respect to the initial configuration, and therefore need to be formed only once if they are stored on the back up storage for use in all the load steps. Further, anisotropy presents no problem in the total Lagrangian formulation [17]. For these reasons, total Lagrangian formulation is adopted in the analysis of a single lap joint.

To analyse a single lap joint, linear formulation can be utilised provided the load is of low magnitude, but it cannot account for large out-of-plane deformations which occur with joint rotations at higher levels of

loading. To accurately analyse a single lap joint a large displacement theory must be used.

Pickett and Hollaway [14,15] have shown the close agreement of peak adhesive stresses with the analytical values when geometrically nonlinear finite element formulation is adopted. The formulation presented here is restricted to large displacement and large rotation, but small strain problems.

In the present paper, adhesively bonded joints have been analysed by modelling the adhesive layer as elasto-viscoplastic and adherend as linearly elastic. It is assumed that the adhesive layer behaves linearly elastic up to initial yield stress and thereafter it enters viscoplastic domain. This facilitates the use of the static yield criterion which separates the elastic domain from the viscoplastic. Behaviour of viscoplastic adhesive material is studied and 3D results are compared with the 2D values to identify 3D zones in the single lap joint. Definition of stress-strain relations, explicit form of Lagrangian formulation and finite element implementation are given in the appendices.

3. Elasto-viscoplastic analysis of solids

3.1. Viscoplastic constitutive law for the adhesive material

The total strain tensor ε_{ij} and consequently $\dot{\varepsilon}_{ij}$, can be decomposed into the elastic and the inelastic (viscoplastic) parts as

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^{vp}, \quad (1)$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^{vp} \quad (2)$$

in which ε_{ij}^e , ε_{ij}^{vp} represent, respectively, the elastic, viscoplastic components, while $\dot{\varepsilon}_{ij}^e$, $\dot{\varepsilon}_{ij}^{vp}$ represent the corresponding strain rate components. In a uniaxial context the viscoplastic element remains inactive when $\sigma < \sigma_y$. The viscoplastic rheological model for uniaxial case is shown in Fig. 1.

The elastic strain is related to the total stress according to the generalized Hook's law,

$$\varepsilon_{ij}^e = \frac{S_{ij}}{2\mu} + \frac{(1-2\nu)}{E} \sigma_m \delta_{ij} \quad (3)$$

in which $S_{ij} = \sigma_{ij} - \delta_{ij}\sigma_m$ is the deviatoric stress tensor, $\sigma_m = 1/3\sigma_{ii}$ is the mean hydrostatic pressure and μ , E , ν are the shear modulus, elastic modulus and Poisson's ratio, respectively.

Consequently, the corresponding elastic strain rate tensor can be written as

$$\dot{\varepsilon}_{ij}^e = \frac{\dot{S}_{ij}}{2\mu} + \frac{(1-2\nu)}{E} \dot{\sigma}_m \delta_{ij}. \quad (4)$$

Viscoplastic behaviour is governed by the scalar yield function of the form

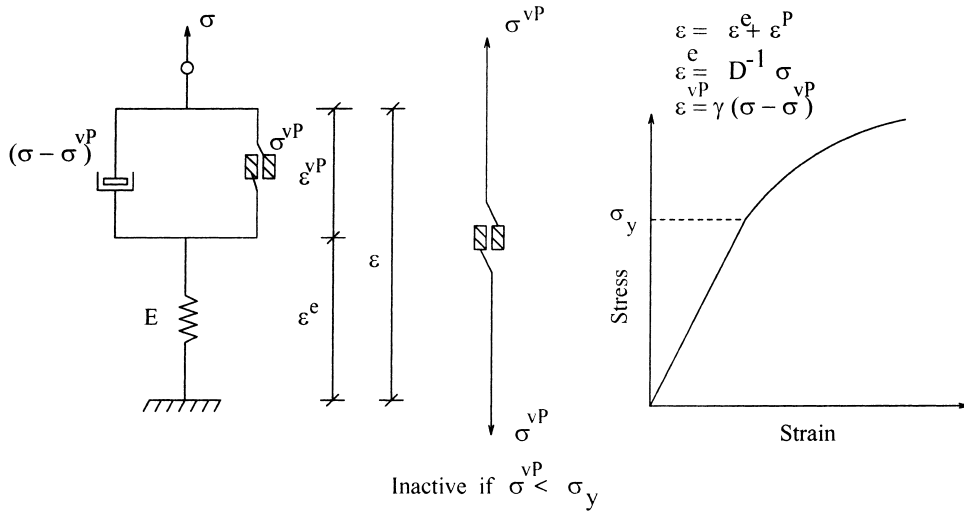


Fig. 1. Uniaxial rheological model showing elasto-viscoplasticity.

$$F = \bar{F}(\sigma_{ij}, \epsilon_{ij}^{vp}, k) - \sigma_y(k) = 0 \quad (5)$$

in which σ_y is the uniaxial or effective yield stress, ϵ_{ij}^{vp} is the viscoplastic strain, k is a history dependent hardening parameter and the value of $F < 0$ implies an elastic state. The viscoplastic strain rate is expressed in its general form as a function of the current stress according to

$$\dot{\epsilon}_{ij}^{vp} = f(\sigma_{ij}). \quad (6)$$

The specific form of the above equation proposed by Perzyna [13] is employed as

$$\dot{\epsilon}_{ij}^{vp} = \gamma \langle \phi(F) \rangle \frac{\partial Q(\sigma_{ij})}{\partial \sigma_{ij}} \quad (7)$$

in which γ is the fluidity parameter controlling the plastic rate, $\phi(F)$ is a positive monotonic increasing flow function, and the notation $\langle \rangle$ implies that the viscoplastic straining occurs only for values of $\phi(F) > 0$. The scalar quantity Q can be interpreted as a plastic potential and for associated viscoplasticity $Q = F$, where F is the yield function. Several choices have been recommended for the function, ϕ , but the two most common forms [8] are

$$\phi(F) = e^{M(\bar{F} - \sigma_y)/\sigma_y} - 1, \quad (8)$$

$$\phi(F) = [(\bar{F} - \sigma_y)/\sigma_y]^N \quad (9)$$

in which M and N are constants. The fluidity parameter γ and the function $\phi(F)$ are determined experimentally.

Finally, the total rate of stress medium can be expressed as

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^{vp}), \quad (10)$$

where C_{ijkl} is the usual constitutive tensor for elastic materials. As the material considered is isotropic, the yield function F is rewritten as

$$F(\sigma_{ij}) = F(\sigma_m, J'_2, J'_3) - \sigma_y(k). \quad (11)$$

In a similar way, Q is also defined, in which σ_m is the mean stress and J'_2, J'_3 are the second and third invariants of the deviatoric stresses, S_{ij} .

3.2. Yield criterion for the adhesive material

Modified von Mises yield criterion suggested by Gali et al. [4] is given by

$$k_a \tau_{oct} + k_v \sigma_m = 1, \quad (12)$$

$$\tau_{oct} = \frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}, \quad (13)$$

$$\sigma_m = \frac{1}{3} [\sigma_1 + \sigma_2 + \sigma_3]. \quad (14)$$

k_a, k_v are the material constants responsible for the yield due to the distortional and isotropic stress components, respectively.

Eq. (12) can be written in terms of stress invariants as

$$F = C_s (J_2)^{1/2} + C_v J_1 - \sigma_y, \quad (15)$$

where $C_s = \sqrt{3}(1 + \lambda)/2\lambda$, $C_v = (\lambda - 1)/2\lambda$ and $\lambda = \sigma_c/\sigma_t$, where J_1 is the first invariant of the general stress and λ is the ratio between compressive and tensile yield stress. The effective strain is given by

$$e = C_s \frac{1}{1 + \nu} (r'_z)^{1/2} + C_v \frac{1}{1 - 2\nu} r'_1, \quad (16)$$

where ν is Poisson's ratio, r'_z is the second invariant of deviatoric strain tensor and r'_1 is the first invariant of the general strain tensor.

Here, in this analysis, the nonlinear stress–strain behaviour of the adhesive material is represented by Ramberg–Osgood equation as well as by a bilinear hardening curve. Ramberg–Osgood curve were obtained by the least square technique and the method is explained in Ref. [11].

4. Finite element model

A single lap joint of adhesive thickness 0.3 mm is modelled using 3D 20 noded isoparametric elements with $3 \times 3 \times 3$ Gaussian integration scheme. Overall dimensions and the boundary conditions of the joint are given in Figs. 2 and 3. The finite element model has 560 elements and 3453 nodes. Adherend material presumed to be linear elastic is intentionally given high initial yield stress value since the focus is on the behaviour of adhesive material. The properties of the materials are listed in Table 1. Viscoplastic analyses are performed by adopting the explicit scheme with the initial time step length equal to 0.1 and time step increment, 1.5. Explicit time scheme is being used instead of implicit scheme since Pandey et al. [12] has shown that the execution time per iteration for implicit scheme is more and the economical and accurate solutions can be obtained by explicit time integration scheme. 3D viscoplastic analysis is performed to look at the stresses in adhesive layer and three different set of results are presented viz.,

Table 1

Material properties of adhesive and adherend [17]

Items	Adherend (steel)	Adhesive (FM-73)
E	21,000.0 MPa	2210.0 MPa
σ_y	5000.0 MPa	40.0 MPa
γ	–	$4.495 \times 10^{-3}/s$
$\phi(F)$	–	$(F - F_0)^N / F_0$
N	–	1.426
λ	–	1.4

- Peel and shear stresses at different time from linear to steady state when the adhesive material is viscoplastic.
- Comparison of stress distribution in the adhesive layer for linear-elastic, elastic solution with geometric nonlinearity, elasto-viscoplastic solution and elasto-plastic solution with the geometric nonlinear effect.
- Identification of 3D zones in the adhesive layer on comparison with the 2D plane strain results of Pandey et al. [11,12].

5. Results and discussions

5.1. Elasto-viscoplastic analysis

Peel and shear stress distribution at the middle layer of the adhesive in the central section of the joint is shown in the Fig. 4. The lap joint is subjected to the tensile load of 200 MPa in one increment and the adhesive material is given elasto-viscoplastic properties. Stresses are redistributed in the adhesive layer till it

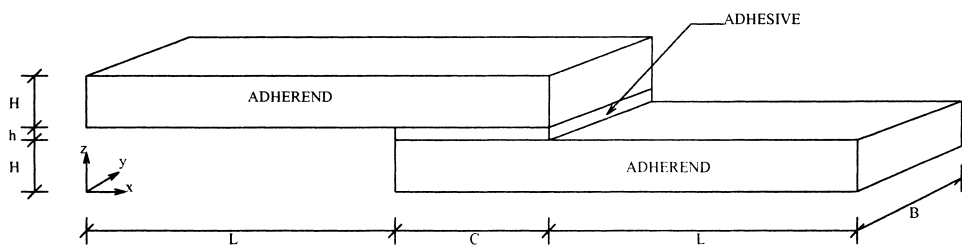


Fig. 2. Single lap joint model with dimensions ($L = 32.0$ mm, $C = 16.0$ mm, $H = 1.6$ mm, $h = 0.3$ mm, $B = 1.0, 4.0, 8.0, 12.0, 16.0, 24.0$ and 32.0 mm).

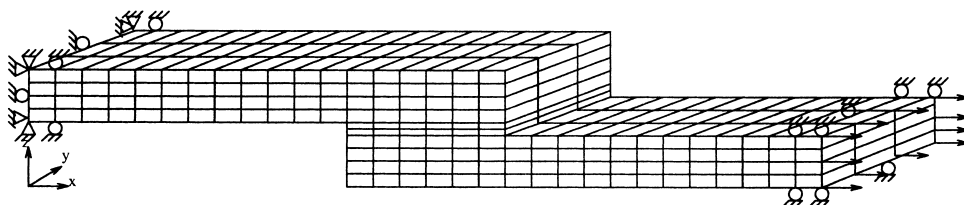


Fig. 3. Typical finite element mesh with boundary conditions.

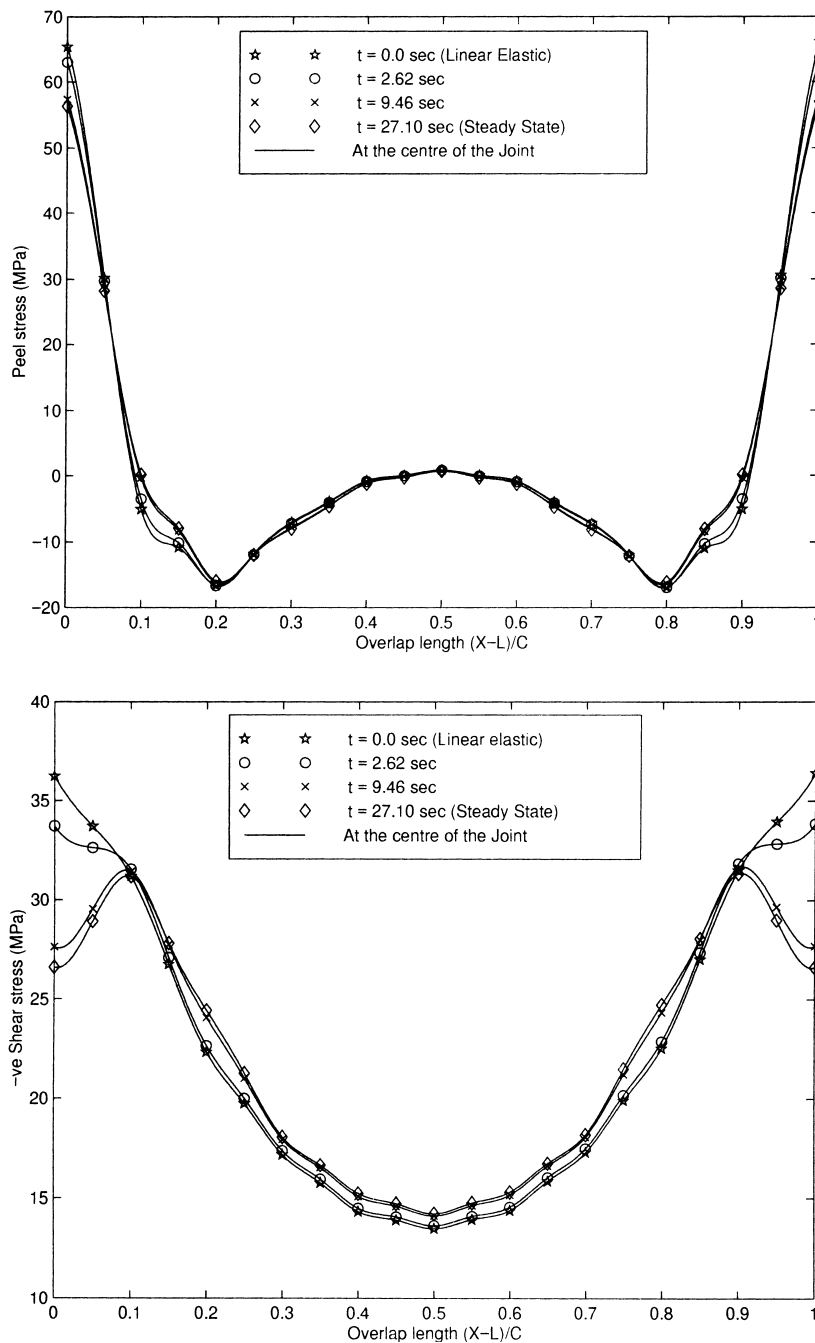


Fig. 4. Peel and shear stress distributions with time at $y = 8.0$ and $z = 1.75$ mm (for $B = 16.0$ mm).

reaches the steady state solution. The stress curves shows the stress variation at different time values starting from time equal to zero which is the linear elastic solution. The steady state solution which corresponds to elastoplastic analysis is reached over the period of 27.1 s

and it took 66 iterations. From the plots, the following points can be deduced:

- Distribution pattern of peel and shear stress is same for different time values.

- Peel stress variation is pronounced at the overlap edges than at the central overlap length.
- Peak peel stress always occurs at the end of overlap length.
- Peak peel stress decreases with time, and peak steady state values is 85% of the linear-elastic solution.
- Peak shear stress decreases with the increase in period and there is a shift in the location of peak stress towards the middle region of overlap length indicating the sensitivity of viscoplastic material with time.
- Steady state peak shear stress is around 72% of the linear stress value.

5.2. Comparison of various analyses results – single lap joint

For linear-elastic and elasto-viscoplastic analysis the load is applied in one increment whereas for geometric nonlinearity, load is applied in 10 increments of equal load factor. The initial yield stress for adhesive and adherend are given larger values so as to remain in elastic region for linear-elastic and elastic solution with geometric nonlinearity. The results of peel and shear stresses for various analyses are shown in the Figs. 5 and 6 and following comparisons are made:

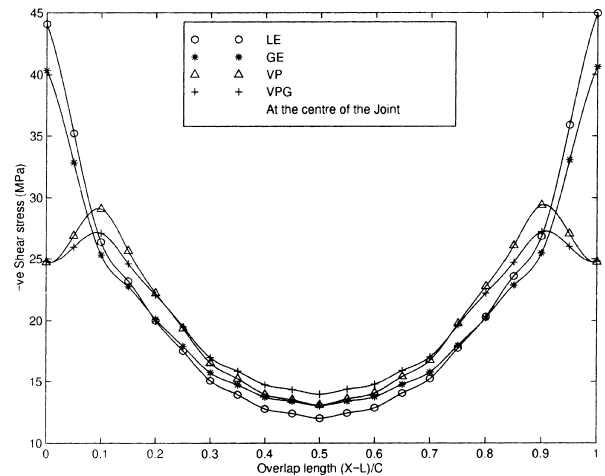
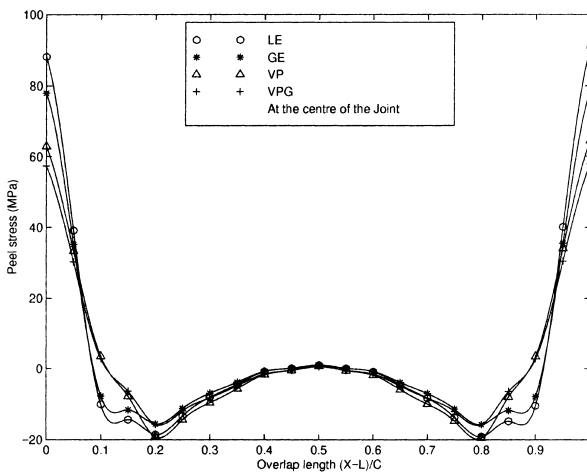


Fig. 5. Peel and shear stress distributions obtained for different types of analysis at $y = 0.5$ and $z = 1.75$ mm (for $B = 1.0$ mm; LE: linear elastic solution; GE: geometric nonlinear elastic solution; VP: viscoplastic solution; VPG: geometric nonlinear viscoplastic solution).

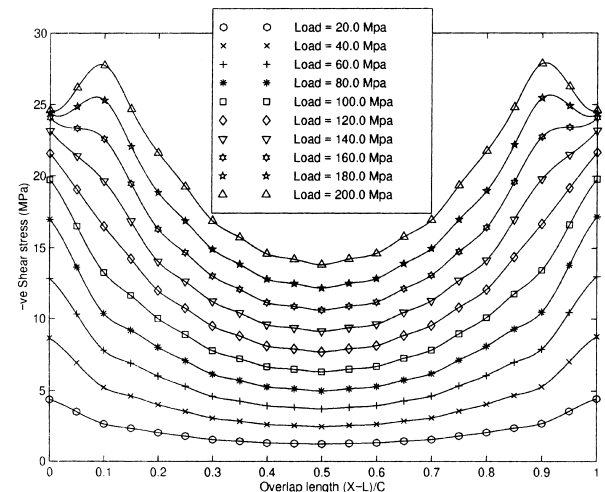
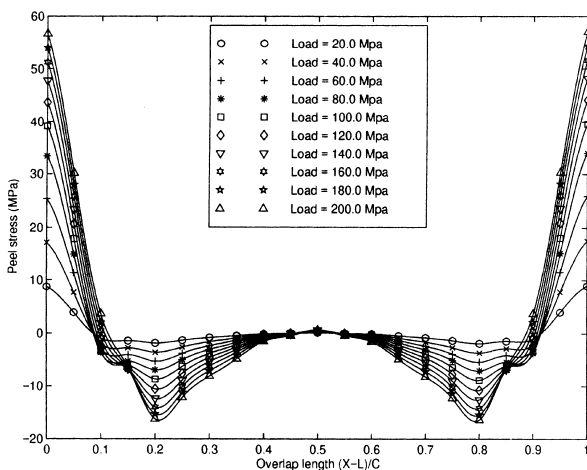


Fig. 6. Peel and shear stress distribution for different load increments.

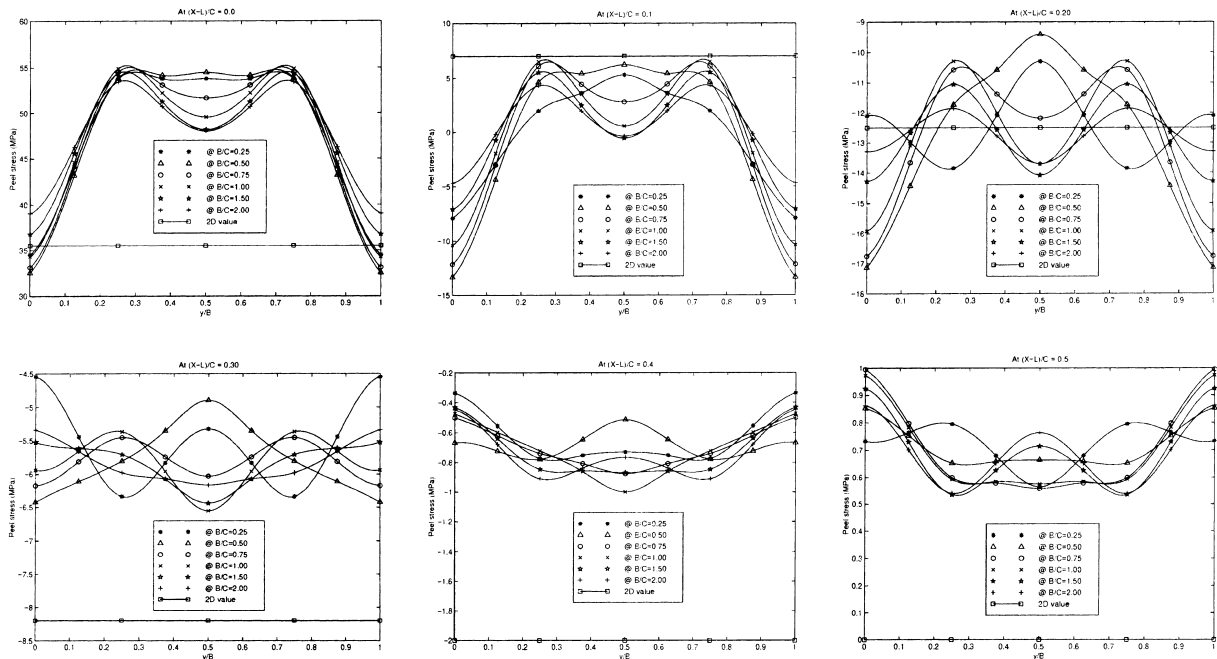


Fig. 7. Peel stress distributions for various B/C ratios at $(X-L)/C = 0.0, 0.1, 0.2, 0.3, 0.4$, and 0.5 .

- Maximum peel stress occurs at the end of the overlap length for all cases.
- Maximum peel stress decreases for the analysis in the order of linear-elastic, elastic with geometric nonlinearity, elasto-viscoplastic and elasto-viscoplastic with geometric nonlinearity.
- Geometric nonlinear analysis reduces the maximum peel and shear stresses at the end portion of overlap for both elastic and elasto-viscoplastic solutions.
- In linear elastic analysis with geometric nonlinearity, peak peel and shear stresses are, respectively, around 80% and 90% of the maximum values obtained without considering geometric nonlinearity.
- Also, in elasto-viscoplastic solution with geometric nonlinearity, peak peel and shear stresses are, respectively, around 90% and 93% of the maximum values obtained without considering geometric nonlinearity.
- From the above two observations, it can be stated that the reduction in peak peel stress is more pronounced than reduction in peak shear stress due to the effect of geometric nonlinearity for both the analysis linear-elastic and elasto-viscoplastic.
- All these observations are in good agreement with the 2D results of Pandey et al. [11].

5.3. Identification of 2D and 3D zones in single lap joint

3D zones in the single lap joint are identified from the zones of plane strain behaviour (i.e. 2D zone) by comparing the 3D results with the 2D results of Pandey et al.

[11,12]. Peel and shear stresses were plotted along the width of the joint at different locations of overlap length. Both the overlap length and the width are made normalized so that the results of different width can be plotted in one graph. The plane strain value [11] will be same throughout the unit width of the joint. This value is picked and plotted as a straight line on the graph of 3D results. Now this will enable us to compare the values at various places along the width of the joint. These plots are given in Figs. 7 and 8. The 3D plots given in Figs. 9 and 10 also shall be referred to identify 3D zones. The observations made from the plots are as follows:

- *Behaviour at the edges:* Peel stress values are close to plane strain [11] at the edges. In particular, the values are matching with the 2D value for the distance of 5% on the ends of lateral width for the ratios $B/C = 0.25, 0.5$ and 0.75 . In the other places, 3D effect is more pronounced. With regard to shear stress, the behaviour is visibly 3D for all the ratios.
- *Behaviour at the centre:* The peel and shear stresses are picked at the centre of the overlap length. On comparison with 2D value, it is clear that the 3D behaviour is enhanced throughout the width, but it should be noted that magnitude is not very significant from 2D value for both the peel and shear stress.
- *Behaviour at other places:* The peel distributions at other places also shows 3D regions clearly from the plane strain locations. But, on the part of shear stress, plane strain value is matching with the stresses

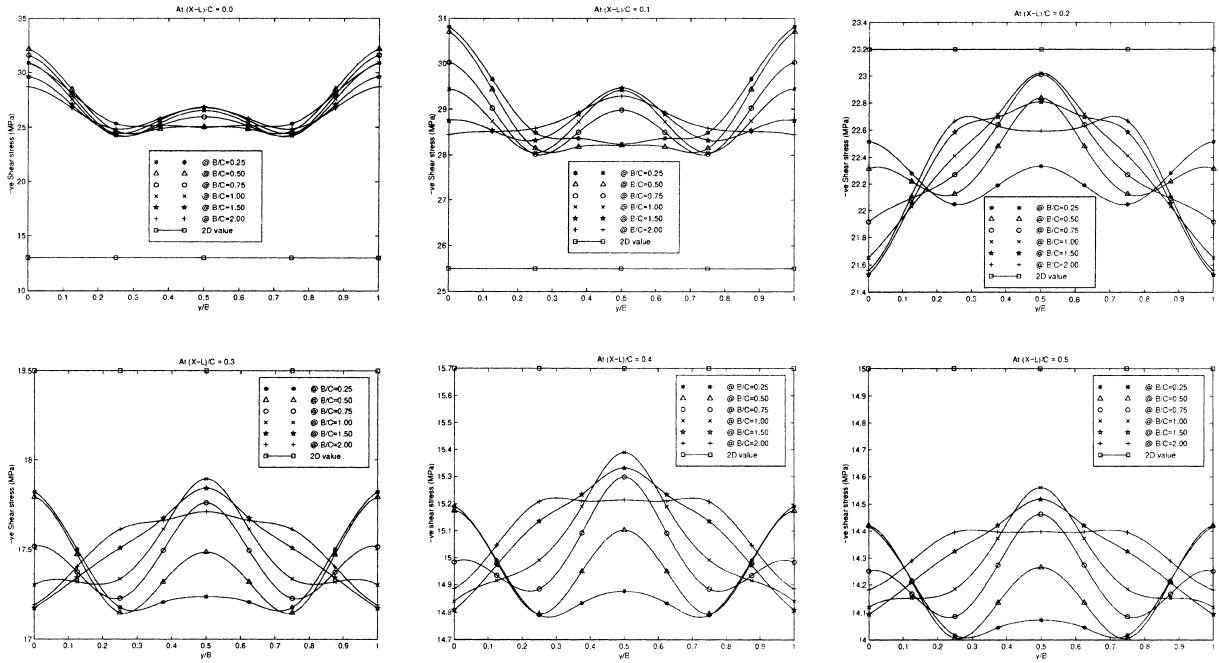


Fig. 8. Shear stress distributions for various B/C ratios at $(X-L)/C = 0.0, 0.1, 0.2, 0.3, 0.4$, and 0.5 .

at the centre of the lateral width and at the overlap locations 10–20% on either side of the centre. It is not same case at 30–40% of the lap length and 3D effect is more pronounced for the full width. Shear stresses too shows the variations but the differences on the magnitude are in the range of 2–6 MPa.

6. Conclusions

3D viscoplastic finite element analysis of adhesively bonded single lap joint is presented. Total Lagrangian method is formulated to consider the geometric non-linearity in the single lap joint due to finite rotation of the joint. The stress–strain of the adhesive material is modelled by the Ramberg–Osgood equation. Modified von Mises criteria is employed for the adhesive material. Viscoplastic analysis gives the reduced stresses at the end of overlap than the elastic solution. From the observations made on the identification of 3D zones, it is concluded that 3D analysis shows significantly different distributions of stresses from the plane strain analysis away from the central region. Hence, the need for 3D analysis is recommended for behavioural study and joint design.

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Appendix A. Method of formulation

A.1. Deformation and strain

The formulation of geometric nonlinearity in this appendix the explicit form of Lagrangian formulation in the next appendix can be referred in detail in Zeinkiewicz [23].

Let $X = [x, y, z]^T$ define the rectangular coordinates of a material point in a body before the deformation. If this point is displaced by $u = [u, v, w]^T$ measured relative to the fixed frame of reference, its new coordinates will become

$$\bar{X} = [\bar{x}, \bar{y}, \bar{z}]^T = X + u. \quad (\text{A.1})$$

Let us consider two neighbouring particles P_0 and Q_0 on the undeformed body. In the final deformed position they are identified by Eulerian coordinates as P and Q . If the distance between P_0 and Q_0 , P and Q are given dS with components $d\bar{X} = [d\bar{x}, d\bar{y}, d\bar{z}]$, respectively. The relation between dX and $d\bar{X}$ is given by

$$\bar{X} = J dX, \quad (\text{A.2})$$

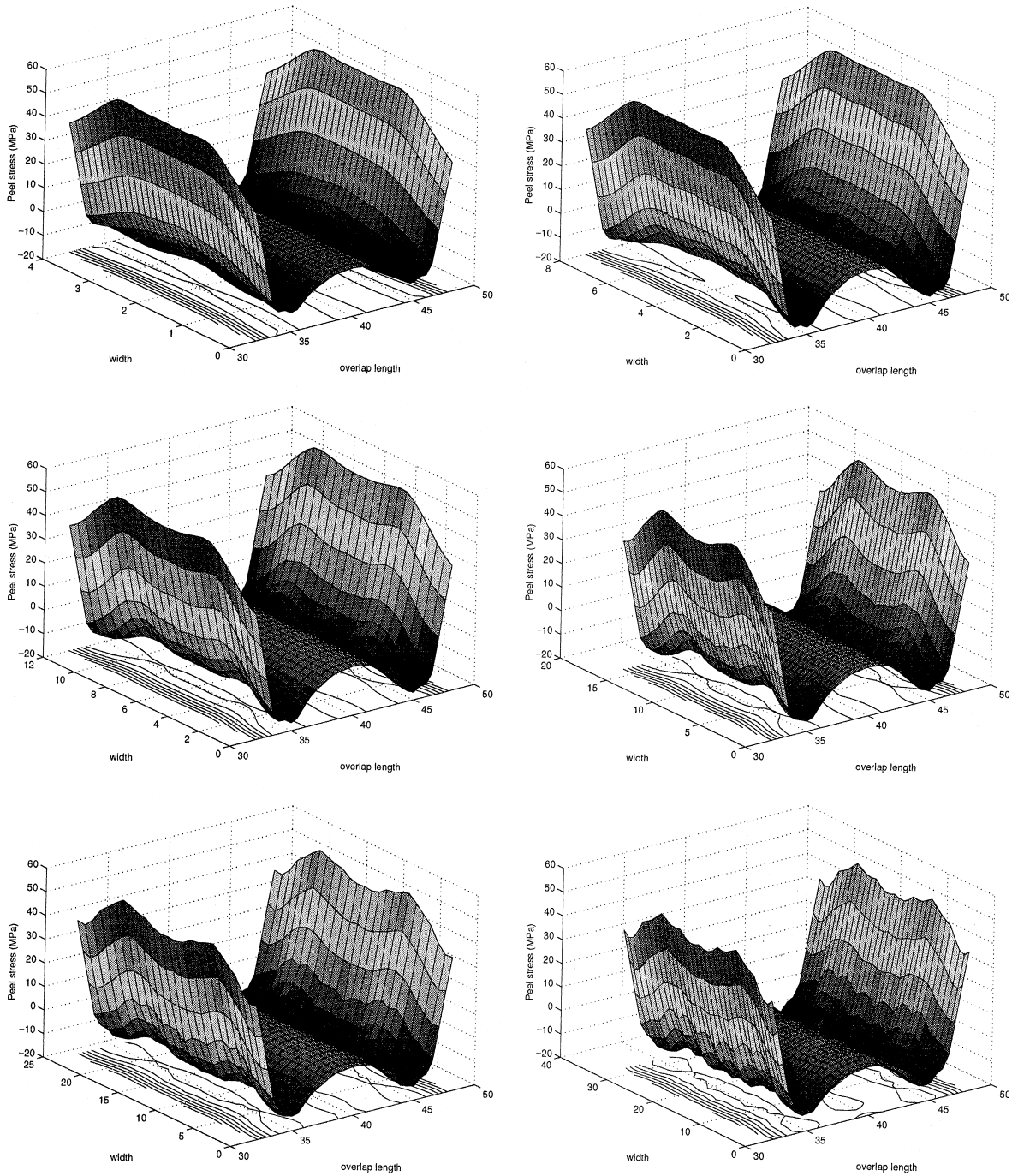


Fig. 9. 3D peel stress distribution for various B/C ratios: 0.25, 0.50, 0.75, 1.00, 1.50, and 2.00.

where

$$J = \begin{bmatrix} \bar{x}, \bar{y}, \bar{z} \\ x, y, z \end{bmatrix} = \begin{bmatrix} \partial \bar{x} / \partial x & \partial \bar{x} / \partial y & \partial \bar{x} / \partial z \\ \partial \bar{y} / \partial x & \partial \bar{y} / \partial y & \partial \bar{y} / \partial z \\ \partial \bar{z} / \partial x & \partial \bar{z} / \partial y & \partial \bar{z} / \partial z \end{bmatrix} \quad (\text{A.3})$$

is a Jacobian matrix defining the deformed state. The change in length can be written as

$$\begin{aligned} \frac{1}{2}(\mathrm{d}\bar{S}^2 - \mathrm{d}S^2) &= \frac{1}{2}(\mathrm{d}\bar{X}^T \mathrm{d}X - \mathrm{d}X^T \mathrm{d}X) = \mathrm{d}X^T \epsilon \mathrm{d}X \\ &= \mathrm{d}\bar{X}^T \epsilon \mathrm{d}\bar{X}, \end{aligned} \quad (\text{A.4})$$

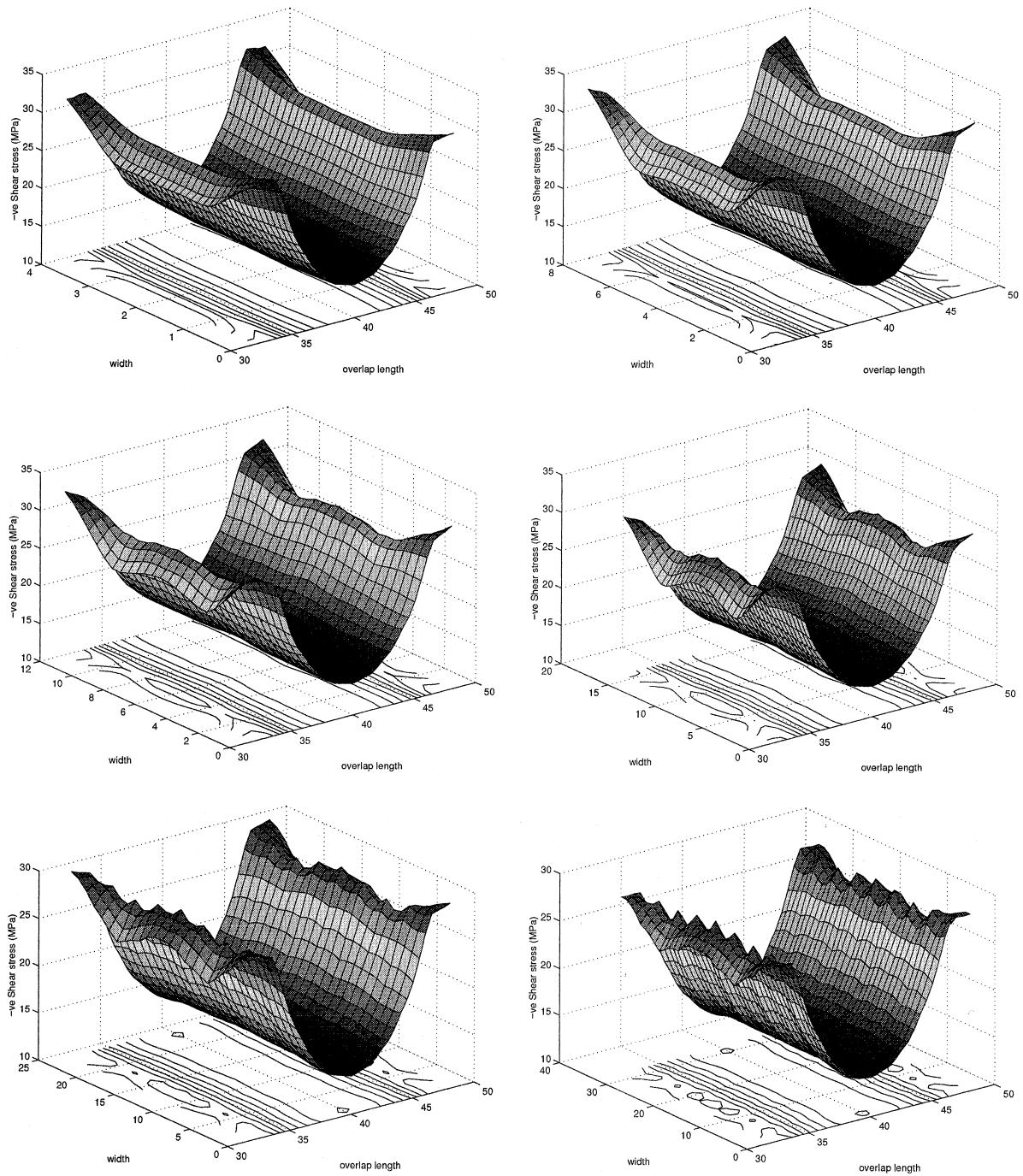


Fig. 10. 3D shear stress distribution for various B/C ratios: 0.25, 0.50, 0.75, 1.00, 1.50 and 2.00.

where

$$\epsilon = \frac{1}{2}(J^T J - I) \quad (\text{A.5})$$

is the Lagrangian definition of the strain (Green's strain) and

$$\epsilon = \frac{1}{2}(I - \bar{J}^T \bar{J}) \quad (\text{A.6})$$

is the Eulerian definition of strain and I is an identity unit matrix.

By substituting Eq. (A.1) into the Eqs. (A.5) and (A.6), the explicit engineering expressions for the strain

components are obtained and they are given in the next section.

A.2. Definitions of stresses

The natural definition of stresses is the Eulerian one referring in the usual way to the forces per unit deformed area. Thus,

$$\bar{\sigma} = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ \bar{\tau}_{yx} & \bar{\sigma}_y & \bar{\tau}_{yz} \\ \bar{\tau}_{zx} & \bar{\tau}_{zy} & \bar{\sigma}_z \end{bmatrix} \quad (\text{A.7})$$

in the matrix notation.

The forces acting on the area $d\bar{A}$ are given in the vector form as

$$d\bar{F} = \bar{\sigma} d\bar{A}. \quad (\text{A.8})$$

As per Hill's [6] definition of conjugate pairs of stress and strain variables, for Green's strain, the conjugate stress is the second Piola–Kirchhoff stress σ . The equivalent force vector acting on an original undeformed area dA is given by an expression:

$$dF = \sigma dA \quad (\text{A.9})$$

and the actual force on the deformed area by

$$d\bar{F} = J dF. \quad (\text{A.10})$$

But the relation between dA and $d\bar{A}$ is given by

$$d\bar{A} = |J| J^{T-1} dA \quad (\text{A.11})$$

or

$$dA = |J|^{-1} J^T d\bar{A}. \quad (\text{A.12})$$

From Eqs. (A.8)–(A.12), we have the stress transformation relation:

$$\sigma = |J| J^{-1} \bar{\sigma} J^{T-1}, \quad (\text{A.13})$$

which shows that the second Piola–Kirchhoff stress is also a symmetric one.

Appendix B. Explicit form of Lagrangian formulation

B.1. Strain–displacement relationships

Green's strain in vector notation in the context of 3D solids can be written using the engineering definitions as

$$\varepsilon_x = \frac{\partial u}{\partial x} + \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right], \quad (\text{B.1})$$

$$\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \quad (\text{B.2})$$

and so on. The strain matrix is given as

$$\epsilon = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}] = \epsilon_0 + \epsilon_1, \quad (\text{B.3})$$

where ϵ_0 is the usual linear, infinitesimal strain vector and ϵ_1 is the nonlinear contribution. The nonlinear part is conveniently written as

$$\epsilon_1 = \frac{1}{2} \begin{bmatrix} \theta_x^T & 0 & 0 \\ 0 & \theta_y^T & 0 \\ 0 & 0 & \theta_z^T \\ \theta_y^T & \theta_x^T & 0 \\ 0 & \theta_z^T & \theta_y^T \\ \theta_z^T & 0 & \theta_x^T \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \frac{1}{2} \underline{A} \theta, \quad (\text{B.4})$$

where

$$\theta_x = [\partial u / \partial x \quad \partial v / \partial x \quad \partial w / \partial x]^T, \quad (\text{B.5})$$

$$\theta_y = [\partial u / \partial y \quad \partial v / \partial y \quad \partial w / \partial y]^T, \quad (\text{B.6})$$

$$\theta_z = [\partial u / \partial z \quad \partial v / \partial z \quad \partial w / \partial z]^T. \quad (\text{B.7})$$

B.2. Nonlinear equilibrium equations

The governing nonlinear equilibrium equations will be established from the virtual work equation, in the Eulerian coordinate system. This is given by the integration over the deformed volume $\bar{V}$ and the deformed area $\bar{A}$ as [24],

$$\int_{\bar{V}} \delta \bar{\epsilon}^T \bar{\sigma} d\bar{V} = \int_{\bar{V}} \bar{\rho} \delta u^T q d\bar{V} + \int_{\bar{A}} \delta u^T \bar{p} d\bar{A} \quad (\text{B.8})$$

in which $\bar{\rho}$ is the density in the deformed state and $\bar{\sigma}$ and $d\bar{\epsilon}$ refer to the vector forms of the Eulerian stress and the deformation increment in the distorted coordinate $\bar{X}$.

Alternatively, the above equation can be written in terms of the variables referred to the original, undistorted, coordinates and now obtain

$$\int_V \delta \epsilon^T \sigma dV = \int_V \rho \delta u^T q dV + \int_A \delta u^T p dA, \quad (\text{B.9})$$

where σ is the second Piola–Kirchhoff stress and $d\epsilon$ is the increment of Green's strain, both referred to the undeformed coordinates.

Then, the relation between ρ and $\bar{\rho}$ [24] is given by

$$\rho = \frac{\bar{\rho}}{|J|}. \quad (\text{B.10})$$

The tractions p defined with respect to the undeformed body are given in terms of the tractions $\bar{p}$ acting over a deformed surface area [19] as

$$p = \left(\frac{d\bar{A}}{dA} \right) \bar{p}, \quad (\text{B.11})$$

where $\bar{p} = p$, for the small strain case with $|J| = 1$ and $(d\bar{A}/dA) = 1$. The displacement u within the element is given as a function of the n nodal displacement as

$$u = N\delta, \quad (\text{B.12})$$

where $\delta = [\delta_1, \delta_2, \dots, \delta_n]$ and N is a shape function matrix with $\delta_i = [u_i, v_i, w_i]$.

After substituting the above expressions for ϵ_0 and differentiating,

$$\begin{aligned} d\epsilon_0 &= B_0 d\delta, \quad \delta_i = [u_i, v_i, w_i]^T, \\ \delta &= [\delta_1, \delta_2, \dots, \delta_n], \end{aligned} \quad (\text{B.13})$$

where B_0 is the small displacement strain matrix and it is given by a typical component sub matrix for the node i as

$$B_0^i = \begin{bmatrix} \partial N_i / \partial x & 0 & 0 \\ 0 & \partial N_i / \partial y & 0 \\ 0 & 0 & \partial N_i / \partial z \\ \partial N_i / \partial y & \partial N_i / \partial x & 0 \\ 0 & \partial N_i / \partial z & \partial N_i / \partial y \\ \partial N_i / \partial z & 0 & \partial N_i / \partial x \end{bmatrix}. \quad (\text{B.14})$$

Differentiation of ϵ_1 yields

$$d\epsilon_1 = \frac{1}{2} d\bar{A}\theta + \frac{1}{2} \bar{A}d\theta, \quad (\text{B.15})$$

which due to the structure of the matrices involved becomes

$$d\epsilon_1 = \bar{A}d\theta. \quad (\text{B.16})$$

Now, the (9×1) vector θ can be written as

$$\theta = Gd = [G1 \dots Gi \dots] \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_i \\ \vdots \end{bmatrix}, \quad (\text{B.17})$$

where

$$G_i = \begin{bmatrix} I_3 & \partial N_i / \partial x \\ I_3 & \partial N_i / \partial y \\ I_3 & \partial N_i / \partial z \end{bmatrix} \quad (\text{B.18})$$

and I_3 is a 3×3 identity matrix.

Substituting Eq. (B.17) into Eq. (B.15), we get

$$d\epsilon_1 = B_1 d\delta \quad (\text{B.19})$$

with

$$B_1 = \bar{A}G. \quad (\text{B.20})$$

Thus, the strain displacement matrix becomes

$$B = B_0 + B_1. \quad (\text{B.21})$$

The total strain matrix in Eq. (B.3) can be written, using Eqs. (B.4), (B.17) and (B.21) as

$$\epsilon = (B_0 + \frac{1}{2}B_1)\delta. \quad (\text{B.22})$$

For the computational purpose, it is convenient to obtain B_1 explicitly by multiplying out the appropriate terms in Eq. (B.20). The expression for the 3D analysis shall be referred in Ref. [24]. From Eq. (B.12) the virtual displacement is

$$du = N d\delta. \quad (\text{B.23})$$

The virtual work Eq. (B.9) is now rewritten as

$$\begin{aligned} d\delta^T \int_V B^T \sigma dV &= d\delta^T \int_V N^T \rho q dV \\ &+ d\delta^T \int_A N^T p dA. \end{aligned} \quad (\text{B.24})$$

Since the virtual displacements $d\delta$ are arbitrary, Eq. (B.24) gives the equilibrium equations in the discretised form as

$$\int_V B^T \sigma dV + f = 0, \quad (\text{B.25})$$

where the vector of equivalent nodal loads f which comprises the vectors due to the body forces f_b , applied tractions f_t , nodal forces f_p , initial stress f_{σ_0} and the initial strain f_{ϵ_0} is given by

$$f = \int_V N^T \rho q dV + \int_A N^T p dA. \quad (\text{B.26})$$

This is identical with that obtained in the infinitesimal displacement case with the crucial exception that B is a linear function of the nodal displacement δ .

Appendix C. Finite element implementation

On the part of establishing equilibrium equations, incremental formulation is implemented in which the static and transient values are updated incrementally corresponding to the successive time steps (or load steps) in order to trace out the complete solution path. In this solution, it is important that the governing finite element equations are satisfied at each time step to sufficient accuracy, otherwise the solution errors can be significantly accumulated that can lead to the solution instabilities. The governing equations of equilibrium which have to be satisfied at any instant of time t_n are given in the form of Eq. (B.25) as

$$\int_V B_n^T \sigma dV + f_n = 0. \quad (\text{C.1})$$

During a time increment $\Delta t = t_{n+1} - t_n$, the equilibrium equations which must be satisfied are given by the incremental form of Eq. (C.1) as

$$\int_V B_n^T \Delta \sigma dV + \int_V dB_n^T \sigma dV + \Delta f_n = 0$$

or

$$\int_V B_n^T \Delta \sigma dV + K_{\sigma n} \Delta d_n + \Delta f_n = 0, \quad (C.2)$$

where Δf_n represents the change in loading occurring during the time interval Δt_n . In majority of the problems encountered in engineering, the load increments are applied in discrete steps and thus $\Delta f_n = 0$ for all the time steps except the first one within an increment. This load increment includes the effect of all the initial strains, body forces and the external boundary load.

The incremental stress change vector occurring in time interval Δt_n is given by

$$\Delta \sigma_n = D \delta \epsilon_n^e = D(\Delta \epsilon_n - \dot{\epsilon}_n^{vp}), \quad (C.3)$$

where D is the elasticity matrix. Using Euler time integration scheme, incremental strain vector between time t_n and $t_{(n+1)}$ is given by

$$\Delta \epsilon_n^{vp} = \left[(1 - \alpha) \dot{\epsilon}_n^{vp} + \alpha \dot{\epsilon}_{n+1}^{vp} \right] \Delta t_n, \quad 0 \leq \alpha \leq 1. \quad (C.4)$$

The incremental stress-strain relationship at time t_n is written as

$$\Delta \sigma_n = \hat{D}_n (B_n \Delta d_n - \dot{\epsilon}_n^{vp} \Delta t_n), \quad (C.5)$$

where

$$\hat{D}_n = (I + D C_n)^{-1} D = (D^{-1} + C_n)^{-1}, \quad (C.6)$$

$$C_n = \alpha \Delta t_n H_n, \quad (C.7)$$

$$H_n = \left[\frac{\partial \dot{\epsilon}^{vp}}{\partial \sigma} \right]_n = H_n(\sigma_n). \quad (C.8)$$

Substitution of $\Delta \sigma$ from Eq. (C.5) for implicit scheme, in the incremental equilibrium Eq. (C.2) gives

$$\Delta d_n = [K_n^T]^{-1} \Delta V_n, \quad (C.9)$$

where the tangential stiffness matrix K_n^T is given by

$$[K_n^T] = K_n + K_{\sigma n} \quad (C.10)$$

with

$$\begin{aligned} K_n &= \int_V B_n^T \hat{D}_n B_n dV, \\ K_{\sigma n} &= \int_V G^T M G dV. \end{aligned} \quad (C.11)$$

The explicit form of M in the above equation is given by Narasimhan [6]. The incremental pseudo loads are given by

$$\Delta V_n = \int_V B_n^T \hat{D}_n \dot{\epsilon}_n^{vp} \Delta t_n dV + \delta f_n + \psi_n \quad (C.12)$$

in which ψ_n represents the vector of residual or out of balance forces given by

$$\psi_n = \int_V B_n^T \sigma_n dV + f_n. \quad (C.13)$$

Thus, for each time step a series of equilibrium iteration cycles should be performed, with the residual forces being applied until they become negligible small. A computationally economic alternative is to avoid the equilibrium iteration process, but to add the residual forces to the pseudo loads to be applied for the next time step. The total displacement and the stresses are computed at time $t_{(n+1)}$ as

$$\delta_{n+1} = \delta_n + \Delta \delta_n, \quad \sigma_{n+1} = \sigma_n + \Delta \sigma_n.$$

Displacement convergence criteria is used with the tolerance limit in the order of 10^{-2} – 10^{-6} , depending upon the desired accuracy. In the viscoplastic algorithm, convergence of the numerical process to the steady state is monitored by comparing the values of the viscoplastic strain rate determined during each time step. The steady state conditions are deemed to be achieved at the end of time step n , if

$$\left[\frac{\Delta t_{n+1} \sum_{G,P} \dot{\epsilon}_{n+1}^{vp}}{\Delta t_1 \sum_{G,P} \dot{\epsilon}_1^{vp}} \right] \leq \text{TOLER}, \quad (C.14)$$

where $\dot{\epsilon}_{n+1}^{vp}$ is the effective strain rate at the end of the n th iteration. TOLER is the specified tolerance.

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