EN222: Mechanics of Solids

Division of Engineering
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2.2 Constitutive laws for plastically deforming solids

Linear elasticity is probably the most commonly used and versatile constitutive law. There are many applications, however, where it is of interest to predict the behavior of solids subjected to large loads, sufficient to cause permanent plastic strains. Examples include:

- Modeling metal forming, machining or other manufacturing processes
- Designing crash resistant vehicles
- Plastic design of structures

Although plasticity theory was developed to predict the behavior of metals under loads exceeding the elastic limit, models rather similar to those developed for metal plasticity are also used to model irreversible damage in various other materials, including microcracking ceramics and concrete, deformation of clay, as well as some polymers.

Here we present a quick crash course on the phenomenological theory of plasticity. We will discuss

1. The general features of the inelastic response of metals (to see what we need to model)
2. Key concepts in modeling inelastic behavior
    a. Decomposition of strain into elastic and plastic parts
    b. Yield criteria
    c. Strain hardening rules
    d. The plastic flow rule
    e. The elastic unloading criterion
3. Summary of constitutive equations for a Mises solid
4. Rate dependent materials and creep

NB *Our initial discussion will focus on nonlinear material behavior, and will assume infinitesimal deformations to keep things simple. We will present the extension to finite deformation plasticity separately.*

2.2.1 Features of the inelastic response of metals.

It is possible to predict the elastic response of a crystal from first principles, using quantum mechanical or more approximate molecular statics computations. This is not the case for inelastic behavior. Although some progress has been made in the past 20 years or so, it is still not possible to predict the plastic stress—strain curve for a crystal. Instead, stress—strain laws for plastically deforming solids are generally fit to experimental measurements. Some fundamental insight into the mechanisms of plastic flow is helpful in guiding the choice of curve fit.

We begin by reviewing the results of a typical tension/compression test on an annealed, ductile, polycrystalline metal specimen (e.g. copper or Al). We assume that the test is conducted at moderate temperature (less than ½ the melting point of the solid – e.g. room temperature) and at modest strains (less than 10%), at modest strain rates ($10^{-4}-10^{-9}$ s^{-1}

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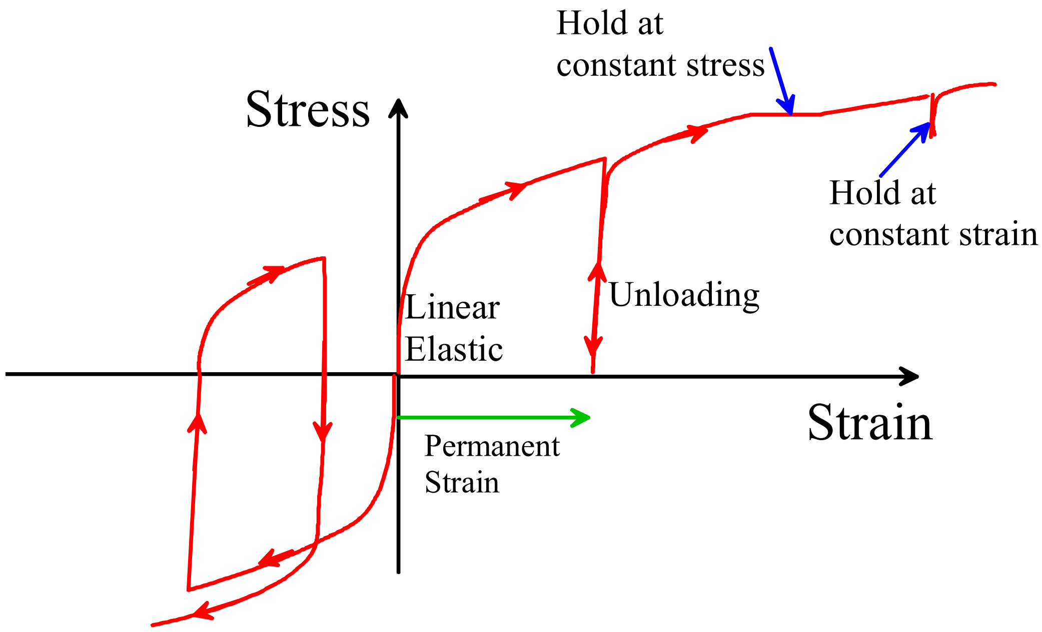


We make the following general observations:

- For modest stresses (and strains) the solid responds elastically. This means the stress is proportional to the strain, and the deformation is reversible.
- If the stress exceeds a critical magnitude, the stress—strain curve ceases to be linear. It is often difficult to identify the critical stress accurately, because the stress strain curve starts to curve rather gradually.
- If the critical stress is exceeded, the specimen is permanently changed in length on unloading.
- If the stress is removed from the specimen during a test, the stress—strain curve during unloading has a slope equal to that of the elastic part of the stress—strain curve. If the specimen is re-loaded, it will initially follow the same curve, until the stress approaches its maximum value during prior loading. At this point, the stress—strain curve once again ceases to be linear, and the specimen is permanently deformed further.
- If the test is interrupted and the specimen is held at constant strain for a period of time, the stress will relax slowly. If the straining is resumed, the specimen will behave as though the solid were unloaded elastically. Similarly, if the specimen is subjected to a constant stress, it will generally continue to deform plastically, although the plastic strain increases very slowly. This phenomenon is known as `creep.'
- If the specimen is deformed in compression, the stress—strain curve is a mirror image of the tensile stress—strain curve (of course, this is only true for modest strains. For large strains, geometry changes will cause differences between the tension and compression tests).
- If the specimen is first deformed in compression, then loaded in tension, it will generally start to deform plastically at a lower tensile stress than an annealed specimen. This phenomenon is known as the `Bauschinger effect.'
- Material response to cyclic loading can be extremely complex. One example is shown in the picture above – in this case, the material hardens cyclically. Other materials may soften.
- The detailed shape of the plastic stress—strain curve depends on the rate of loading, and also on temperature. In general, higher strain rates require higher stresses, and higher temperatures reduce the stress.

We also need to characterize the multi-axial response of an inelastic solid. This is a much more difficult experiment to do. Some of the nicest experiments were done by G.I. Taylor and collaborators in the early part of the last century. Their approach was to measure the response of thin-walled tubes under combined torsion and axial loading.

The main conclusions of these tests were

- The shape of the uniaxial stress-strain curve is insensitive to hydrostatic pressure. The ductility (strain to failure) is increased by hydrostatic pressure, particularly under torsional loading.
- Plastic strains are volume preserving, i.e. $d\varepsilon_{kk} = 0$
- During plastic loading, the principal components of the plastic strain tensor are parallel to the components of stress acting on the solid. This sounds obvious until you think about it… Suppose you were to take a cylindrical shaft and pull it until it starts to deform plastically. Then, holding the axial stress fixed, apply a torque to the shaft. Experiments show that the shaft will initially *stretch*, rather than rotate. The plastic strain increment is proportional to the stress acting on the shaft, *not the stress increment*. This is totally unlike elastic deformation.
- Under multi-axial loading, most annealed polycrystalline solids obey the *Levy-Mises flow rule*, which relates the principal components of strain increment during plastic loading to the principal stresses as follows

$$\frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_1 - d\varepsilon_3}{\sigma_1 - \sigma_3} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3}$$

We proceed to develop a constitutive law that will capture these features of material behavior. For simplicity, we will at this stage ***restrict attention to infinitesimal deformations.***

Consequently, we adopt the infinitesimal strain tensor

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

as our deformation measure. We have no need to distinguish between the various stress measures and will use σ_{ij} to denote stress.

Key ideas in modeling metal plasticity

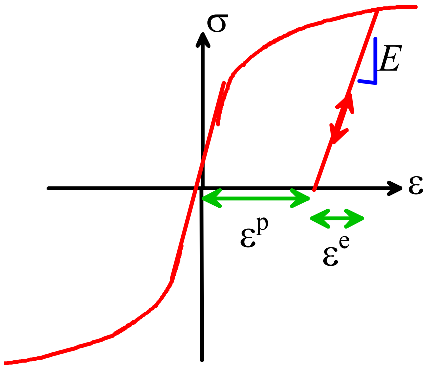
Three key concepts form the basis of almost all classical theories of plasticity. They are

1. The decomposition of strain into elastic and plastic parts
2. Yield criteria: which predict whether the solid responds elastically or plastically
3. Strain hardening rules, which control the shape of the stress-strain curve in the plastic regime;
4. The plastic flow rule, which determines the relationship between stress and plastic strain under multi-axial loading;
5. The elastic unloading criterion, which models the irreversible behavior

We proceed to discuss these concepts in turn.

2.2.2. Decomposition of strain into elastic and plastic parts

Experiments show that under uniaxial loading, the strain at a given stress has two parts: a small recoverable elastic strain, and a large, irreversible plastic strain.



$$\varepsilon = \varepsilon^e + \varepsilon^p$$

Experiments suggest that the reversible part is related to the stress through the usual linear elastic equations. Plasticity theory is concerned with characterizing the irreversible part.

For multiaxial loading, we generalize this by decomposing a general strain increment $d\varepsilon_{ij}$ into elastic and plastic parts, as

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$$

The elastic part of the strain is related to stress using the linear elastic equations

$$C_{ijkl} d\varepsilon_{kl}^e = d\sigma_{ij}$$

2.2.3. Yield criteria

Before attempting to develop a full description that relates plastic strain to stress, it is sensible to establish criteria that will predict the onset of inelastic deformation.

The general nature of a yield criterion for a polycrystalline solid can be argued out. We make two assumptions

(a) Yield is independent of hydrostatic pressure;
(b) The solid is isotropic

Assumption (a) implies that the yield criterion can only depend on the *deviatoric* stress components

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$$

Assumption (b) implies that the onset of yield can only depend on the magnitudes of the principal stresses S_1, S_2, S_3, and can't depend on the principal stress directions. Another way to state the same thing is to note that yield can only depend on the *invariants* of the deviatoric stress tensor

$$J_1 = 0$$

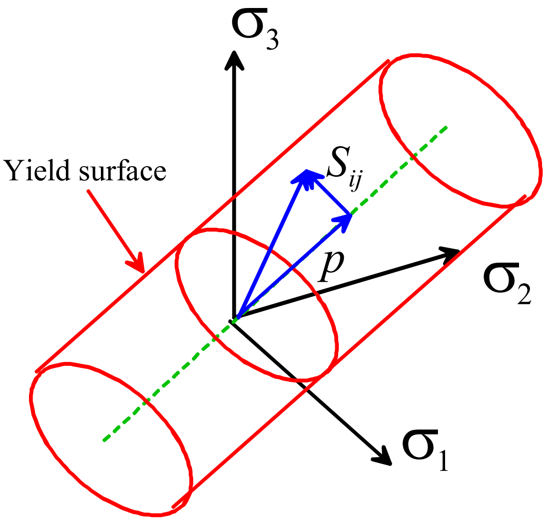
$$J_2 = \frac{1}{2} S_{ij} S_{ij}$$

$$J_3 = \frac{1}{3} S_{ik} S_{kj} S_{ji}$$

Note that we are using *J* to denote the invariants to distinguish them from the conventional *I* – partly to emphasize that these are invariants of the *deviatoric* stress, and partly to emphasize that the definitions here are slightly different to the conventional form (just to make things look neater). The conditions necessary to cause yield could be expressed as or .

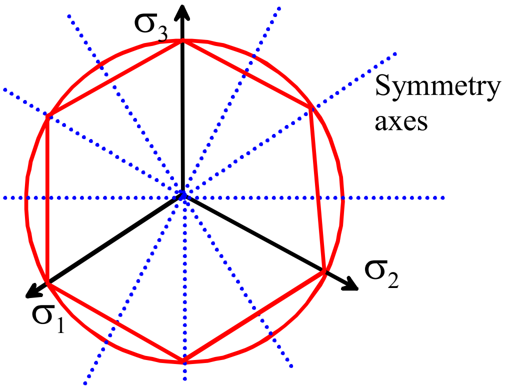
make things look neater). The conditions necessary to cause yield could be expressed as $f(\sigma_1,\sigma_2,\sigma_3,\text{material state variables})=0$ or $F(J_2,J_3,\text{material state variables})=0$, where `material state variables' somehow characterize the strength of the solid – they could include a strain history dependent yield stress and temperature, for example.

Another way to visualize yield criteria is to represent a stress state in principal stress state. Since the material response can only depend on $\sigma_1,\sigma_2,\sigma_3$, we can represent all loading conditions as a point in 3D space with $\sigma_1,\sigma_2,\sigma_3$ as the axes.



A little thought shows that in this space, the yield criterion must look like a (not necessarily circular) cylinder with its axis parallel to the line $\sigma_1=\sigma_2=\sigma_3$ This is because a hydrostatic stress state always falls on this line. A general stress state therefore consists of a hydrostatic component parallel to the line, and a deviatoric component, which must be perpendicular to the line (by definition, deviatoric stress has no hydrostatic component). The solid yields when the deviatoric stress reaches a critical magnitude – this condition defines the cylinder shown.

This is a nice observation because it allows us now to reduce the yield criterion to a 2D representation. If we look down the axis of the cylinder, we would see a view like the one shown below



Of course the actual yield surface need not be circular. As far as yield is concerned, this picture says it all.

We can now argue out several more features of the yield surface. For an isotropic solid, we can freely interchange $\sigma_1, \sigma_2, \sigma_3$ without affecting yield. This means that the yield surface must have 6 axes of symmetry as shown in the picture.

There is also good reason to believe that for most materials the yield surface as represented in principal stress space must be convex. This is not obvious and requires some insight into the mechanisms of plastic flow – we will return to this question later.

If the yield surface is convex, however, it must lie somewhere between the hexagonal and circular surfaces shown in the picture. These two limiting surfaces have names: the hexagon may be expressed as

Tresca yield criterion is the hexagonal surface

$$\max\left\{|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\right\} - Y = 0$$

$$4(J_2 - k)(J_2 - 4k^2)^2 - 27J_3^2 = 0$$

where *Y* and *k* are material constants (specifying the size of the yield locus) and represent the yield stress of the solid in uniaxial tension and simple shear, respectively. For the Tresca surface they are related by *Y=2k*

Von-Mises yield criterion is the circular surface, which can be expressed in various ways

$$\sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2\right]} - Y = 0$$

$$\sqrt{\frac{3}{2} S_{ij} S_{ij}} - Y = 0$$

$$\sigma_e - Y = 0$$

$$\sqrt{J_2} - k = 0$$

where *Y* and *k* have the same meaning as before, and $\sigma_e = \sqrt{3S_{ij}S_{ij}/2}$ is the Von-Mises effective stress. For this yield surface the two constants are related by $Y = \sqrt{3}k$

Note that the two possible extremes of the yield surface differ very little. It is difficult to distinguish between them experimentally.

2.2.4. Strain hardening laws

Experiments show that if you plastically deform a solid, then unload it, and then try to re-load it so as to induce further plastic flow, its resistance to plastic flow will have increased. This is known as strain hardening.

Obviously, we can model strain hardening by relating the size and shape of the yield surface to plastic strain in some appropriate way. There are many ways to do this, of course. Here we describe the simplest approaches.

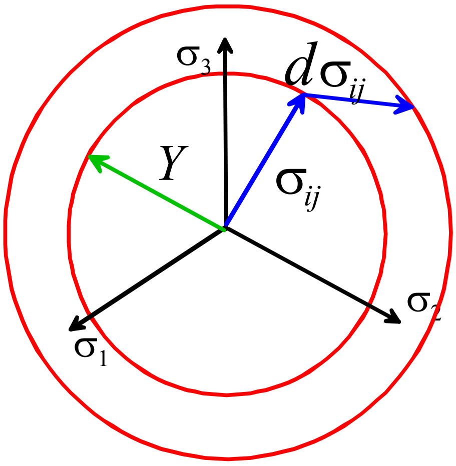
Isotropic hardening

Rather obviously, the easiest way to model strain hardening is to make the yield surface increase in size, but remain the same shape, as a result of plastic straining.

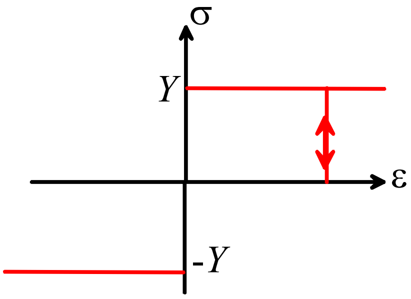
This means we come up with some appropriate relationship between Y or k and the plastic strain. To get a suitable scalar measure of plastic strain we can use either the accumulated plastic work

$W = \int \sigma_{ij} d\varepsilon_{ij}^{p}$ or the accumulated plastic strain magnitude $\lambda = \int \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}}$ (the factor of 2/3 is introduced so that $\lambda = \varepsilon$ under a uniaxial strain ε).

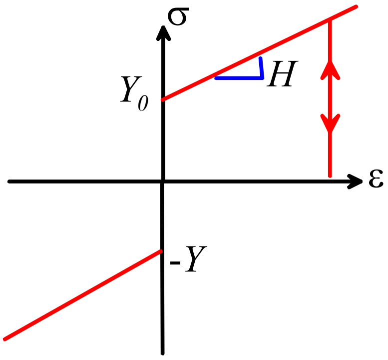
Then we put $Y = H(\lambda)$ or an equivalent expression involving W and/or k. The function H can be determined by fitting a curve to a uniaxial tension test. People often use power laws or piecewise linear approximations in practice.



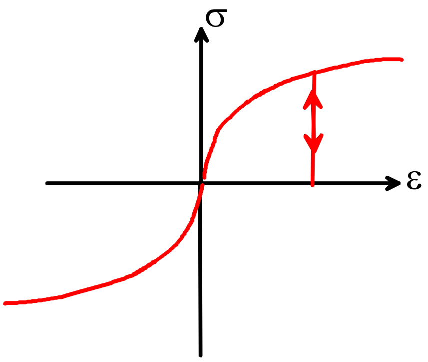
A few of the more common forms of hardening functions are illustrated below



Perfectly plastic solid



Linear strain hardening solid

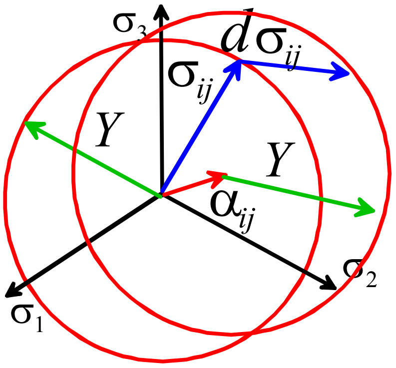


Power-law hardening solid

Kinematic hardening

An isotropic hardening law is generally not useful in situations where components are subjected to cyclic loading. It does not account for the Bauschinger effect, and so predicts that after a few cycles the solid will just harden up until it responds elastically.

To fix this, an alternative hardening law allows the yield surface to translate, without changing its shape. The idea is illustrated graphically below. As you deform the material in tension, you drag the yield surface in the direction of increasing stress, thus modeling strain hardening. This *softens* the material in compression, however. So, this constitutive law can model cyclic plastic deformation.



To account for the fact that the center of the yield locus is at a position α_{ij} in stress space, we would re-write the Von Mises yield criterion as

$$\sqrt{\frac{3}{2}\left(S_{ij}-\alpha_{ij}\right)\left(S_{ij}-\alpha_{ij}\right)}-Y=0$$

To model hardening, we need to relate α_{ij} to the plastic strain history somehow. There are many sneaky ways to do this, which can model subtle features of the plastic response of solids under cyclic and nonproportional loading.

The simplest approach is to set $d\alpha_{ij} = c\,d\varepsilon_{ij}^{p}$. This evidently predicts that the stress-plastic strain curve is a straight line with slope c. This is known as *linear kinematic hardening*

A more sophisticated approach sets

$$d\alpha_{ij} = c\,d\varepsilon_{ij}^{p} - \gamma\alpha_{ij}\,d\lambda$$

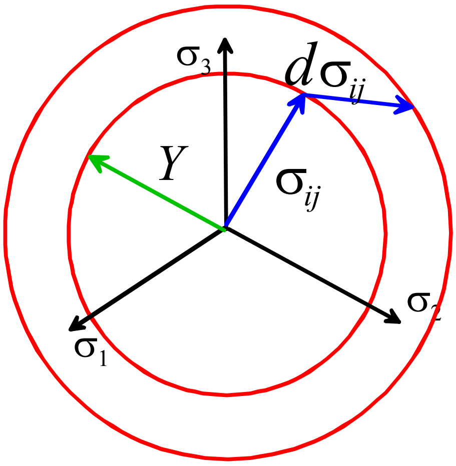
where c and γ are material constants. It's not so easy to visualize what this does – it turns out that that this relation can model *cyclic creep* – the tendency of a material to accumulate strain in the direction of mean stress under cyclic loading. It is known as the Armstrong-Frederick hardening law.

There are many others. Some use multiple nested yield surfaces. New ones are still being developed. You can develop your own!

2.2.5. The plastic flow rule

To complete our constitutive model, we need to devise a way to calculate the plastic strains induced by loading beyond yield.

To make this precise, consider first applying a stress σ_{ij} that is just sufficient to reach yield. Now, increase the stress to $\sigma_{ij} + d\sigma_{ij}$. What is the resulting plastic strain increment $d\varepsilon_{ij}$?



Plastic strain magnitude – the consistency condition

Note first that the *magnitude* of the plastic strain is completely determined by the hardening behavior of the solid. This is because during continued plastic flow, the stress must be on the yield surface at all times. Since the radius (or position, for kinematic hardening) of the yield surface is related to the magnitude of the plastic strain increment through an appropriate hardening law, this means the plastic strain magnitude must be related to the stress increment.

To be precise, let $d\lambda = \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}}$ denote the magnitude of the plastic strain increment (the factor 2/3 is introduced so that $d\lambda = d\varepsilon$ under a uniaxial strain). In addition, let $f\left(\sigma_{ij}, \lambda\right) = 0$ denote the yield criterion . Then, the condition that the solid is at yield during plastic straining gives

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{df}{d\lambda} d\lambda = 0$$

This is known as the consistency condition. Then

$$d\lambda = \frac{1}{h} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$

where $h = -\frac{\partial f}{\partial \lambda}$ is the slope of the stress-plastic strain curve under uniaxial loading.

As an example, let us apply this procedure to an isotropically hardening Mises solid. In this case,

$$f = \sqrt{\frac{3}{2} S_{ij} S_{ij}} - H(\lambda) = 0$$

giving

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial S_{kl}} \frac{\partial S_{kl}}{\partial \sigma_{ij}} = \left(\sqrt{\frac{3}{2}} \frac{S_{kl}}{\sqrt{S_{pq} S_{pq}}} \right) \left(\delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{im} \delta_{jm} \delta_{kl} \right) = \frac{3}{2} \frac{S_{ij}}{\sqrt{\frac{3}{2} S_{kl} S_{kl}}}$$

$\frac{\partial f}{\partial \lambda} = -\frac{dH}{d\lambda} = -h$, where h is the slope of the uniaxial stress – plastic strain curve. Hence, for this case

$$d\lambda = \frac{1}{h} \frac{\frac{3}{2} S_{ij} d\sigma_{ij}}{\sqrt{\frac{3}{2} S_{ij} S_{ij}}}$$

Plastic strain direction - the flow rule

We need therefore to specify only the ratios of the plastic strain components. The most general relationship must have the form

$$d\varepsilon_{ij}^{p} = d\lambda G_{ij}$$

where G_{ij} may be a function of stress and strain history, and must satisfy

$G_{ij} = G_{ji}, \quad G_{ij} G_{ij} = 3/2$ (the latter condition ensures that $d\lambda = \sqrt{2/3 d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}}$)

Further restrictions on G_{ij} follow from three experimental observations:

1. Plastic strains are volume preserving (to a close approximation) so $G_{ii} = 0$
2. Plastic strain increments for crystalline solids are independent of hydrostatic loading
3. Many annealed polycrystalline solids are isotropic, at least for small strains.

A constitutive law with these features may be constructed by setting

$$G_{ij} = \frac{\partial g}{\partial \sigma_{ij}}$$

where the function

$$g = g(J_2, J_3, \text{strain history,temperature})$$

is known as the *plastic flow potential* for the solid.

Observe that although the plastic strain magnitude depends on $d\sigma_{ij}$, the ratios of the plastic strain components depend on *current stress*, not the increment in stress.

The structure of *g* is evidently very similar to the yield function *f*. In particular, in stress space it must be a cylinder, with the same symmetries as *f*.

Without further experimental data we cannot specify *g* further. However, we find that our constitutive law has a mathematically pleasing structure if we make the plastic flow potential the same shape as the yield surface. Moreover, this approximation closely approximates experimental observations, and also has some justification from dislocation based theories of plasticity (see the next section). A flow law derived from the yield surface is known as an *associated plastic flow law*. Theories of plasticity that use a separate flow potential are known as *non-associative* plasticity models.

Specifically, for an associated plasticity model with yield surface

$$f(\sigma_{ij}) = 0$$

we set

$$g(\sigma_{ij}) = Cf(\sigma_{ij})$$

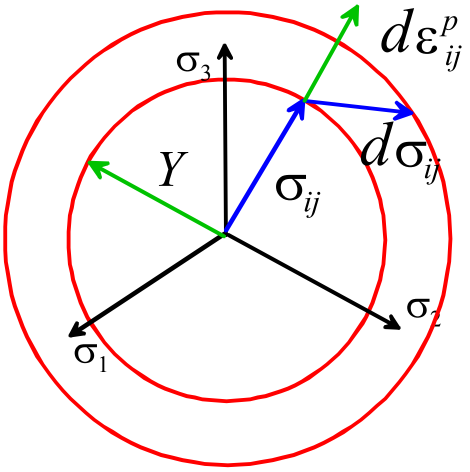
where the constant C is chosen so that

$$C^2 \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2}$$

to ensure that $\sqrt{(2/3)d\varepsilon_{ij}^p d\varepsilon_{ij}^p} = d\lambda$ as before. This then gives

$$d\varepsilon_{ij}^p = d\lambda C \frac{\partial f}{\partial \sigma_{ij}}$$

The associated flow law has a simple graphical interpretation. Observe that since $f(\sigma_{ij}) = \text{constant}$ on the yield surface then $\partial f / \partial \sigma_{ij}$ must be perpendicular to the surface (the gradient of a function is perpendicular to its level contours). Thus, the plastic strain vector represented in 2D stress space is normal to the yield locus, as shown below



Associated flow law for Mises solid

As an example, consider a material with an isotropically hardening Von-Mises yield function

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}} - H(\lambda) = 0$$

where $S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

In this case we already calculated that

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2}\frac{S_{ij}}{\sqrt{\frac{3}{2}S_{kl}S_{kl}}}$$

whence

$$d\varepsilon_{ij}^{p} = d\lambda \frac{3}{2}\frac{S_{ij}}{\sqrt{\frac{3}{2}S_{kl}S_{kl}}}$$

This is known as the Levy-Mises flow rule.

This flow law is sometimes expressed in terms of principal stresses and strains. For principal stresses $\sigma_1, \sigma_2, \sigma_3$ we have that

$$d\varepsilon_1 - d\varepsilon_2 = d\lambda\sqrt{\frac{3}{2}}\frac{S_1 - S_2}{\sqrt{S_1^2 + S_2^2 + S_3^2}} = d\lambda\sqrt{\frac{3}{2}}\frac{\sigma_1 - \sigma_2}{\sqrt{S_1^2 + S_2^2 + S_3^2}}$$
$$d\varepsilon_1 - d\varepsilon_3 = d\lambda\sqrt{\frac{3}{2}}\frac{S_1 - S_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}} = d\lambda\sqrt{\frac{3}{2}}\frac{\sigma_1 - \sigma_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}}$$

which implies that

$$\frac{d\varepsilon_1 - d\varepsilon_2}{d\varepsilon_1 - d\varepsilon_3} = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}$$

and similarly for other components.

Application to other hardening laws

To derive relationships between plastic strain increment and stress increment for a more general hardening law (e.g. the kinematic hardening law) we write the yield function as

$$f(\sigma_{ij}, q_i) = 0$$

where q_i are a set of state variables (e.g. α_{ij}, or other parameters that characterize the material state). The hardening law must specify a relationship between dq_i and plastic strain increment $d\varepsilon_{ij}^{p}$ for each state variable – this must be established by experiment. Then, the consistency condition requires that

$$\frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial f}{dq_i}\frac{\partial q_i}{\partial \varepsilon_{kl}}d\varepsilon_{kl}^{p} = 0$$

while the flow rule requires

$$d\varepsilon_{ij}^{p} = Cd\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

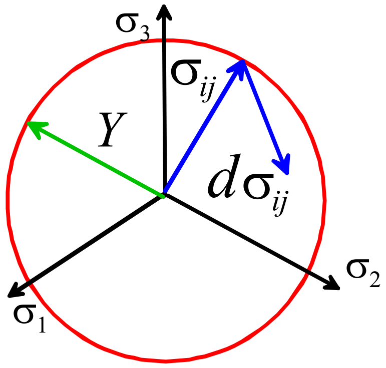
so combining these to

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{dq_i}\frac{\partial q_i}{\partial \varepsilon_{kl}}\frac{\partial f}{\partial \sigma_{kl}} Cd\lambda = 0$$

we can solve for the unknown factor $Cd\lambda$ in the flow law (there is no need to solve for C and $d\lambda$ separately).

2.2.6. Elastic unloading condition

There is one final issue to consider. Experiments show that plastic flow is irreversible, and always dissipates energy. If the increment in stress $d\sigma_{ij}$ is tangent to the yield surface, or brings the stress below yield, then it induces no plastic strain.



For an isotropically hardening solid, this condition may be expressed as

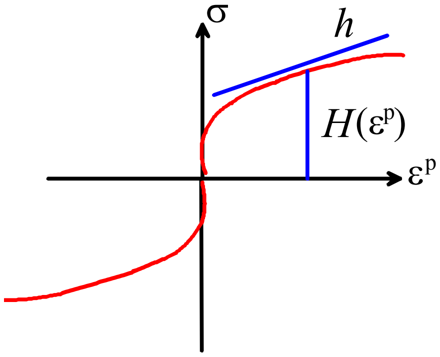
$$S_{ij} d\varepsilon_{ij}^{p} \geq 0$$

A similar condition may be derived for kinematic hardening (can you see what it is?)

Complete stress-strain relations for a rate independent isotropic Mises solid

We conclude by summarizing the complete elastic-plastic stress strain relations for an isotropic Mises solid.

The solid is characterized by its elastic constants μ, ν and by the stress-plastic strain curve under uniaxial loading, which is fit by a curve $H(\lambda)$ with slope $h = \frac{dH}{d\lambda}$ (the is also a function of λ)



In this case we have that

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

with

$$d\varepsilon_{ij}^{e} = \frac{1}{2\mu}\left(d\sigma_{ij} - \frac{\nu}{1+\nu} d\sigma_{kk}\delta_{ij}\right)$$

$$d\varepsilon_{ij}^{p} = \begin{cases} 0 & \sqrt{\frac{3}{2}S_{ij}S_{ij}} - H(\lambda) < 0 \\ \left\langle \frac{3}{2}\frac{S_{kl}d\sigma_{kl}}{Hh} \right\rangle \frac{3}{2}\frac{S_{ij}}{H} & \sqrt{\frac{3}{2}S_{ij}S_{ij}} - H(\lambda) = 0 \end{cases}$$

where $\langle x \rangle = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

These may be combined to

$$d\varepsilon_{ij} = \begin{cases} \frac{1}{2\mu}\left(d\sigma_{ij} - \frac{\nu}{1+\nu} d\sigma_{kk}\delta_{ij}\right) & \sqrt{\frac{3}{2}S_{ij}S_{ij}} - H(\lambda) < 0 \\ \frac{1}{2\mu}\left(d\sigma_{ij} - \frac{\nu}{1+\nu} d\sigma_{kk}\delta_{ij}\right) + \left\langle \frac{3}{2}\frac{S_{kl}d\sigma_{kl}}{Hh} \right\rangle \frac{3}{2}\frac{S_{ij}}{H} & \sqrt{\frac{3}{2}S_{ij}S_{ij}} - H(\lambda) = 0 \end{cases}$$

It is sometimes necessary to invert these expressions. This is straightforward.

First, take care of the hydrostatic terms by contracting the strain tensor

$$d\varepsilon_{kk} = \frac{1}{2\mu}\frac{1-2\nu}{1+\nu} d\sigma_{kk} \qquad \Rightarrow d\sigma_{kk} = \frac{2\mu(1+\nu)}{1-2\nu} d\varepsilon_{kk}$$

Next, note that at yield

$$S_{ij}d\varepsilon_{ij} = \frac{1}{2\mu}\left(S_{ij}d\sigma_{ij}\right) + \left\langle \frac{3}{2}\frac{S_{kl}d\sigma_{kl}}{Hh} \right\rangle \frac{3}{2}\frac{S_{ij}S_{ij}}{H}$$

$$\frac{2}{3}S_{ij}S_{ij} - H^2 = 0$$

$$\Rightarrow S_{ij}d\sigma_{ij} = \frac{2\mu h}{3\mu + h}S_{ij}d\varepsilon_{ij}$$

Substituting back into the stress-strain equations and accounting for elastic unloading, we find that the inverted stress-strain relations become

$$d\sigma_{ij} = \begin{cases} 2\mu\left\{ d\varepsilon_{ij} + \dfrac{\nu}{1-2\nu}d\varepsilon_{kk}\delta_{ij} \right\} & \sqrt{\dfrac{3}{2}S_{ij}S_{ij}} - H(\lambda) < 0 \\ 2\mu\left\{ d\varepsilon_{ij} + \dfrac{\nu}{1-2\nu}d\varepsilon_{kk}\delta_{ij} - \dfrac{3\mu}{3\mu+h}\left\langle \dfrac{3}{2}\dfrac{S_{kl}d\varepsilon_{kl}}{H} \right\rangle \dfrac{3}{2}\dfrac{S_{ij}}{H} \right\} & \sqrt{\dfrac{3}{2}S_{ij}S_{ij}} - H(\lambda) = 0 \end{cases}$$

Similar expressions can be derived for other yield surfaces and hardening laws.

2.2.7. Rate dependent solids and creep

The ideas outlined in the preceding sections can easily be extended to model rate dependent and creeping solids. The goal in modeling creep is to predict the steady accumulation of plastic strain at constant stress.

In this case we decompose the strain *rate* into elastic and plastic parts as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p}$$

The elastic strain rate is given by the usual linear elastic relations. The plastic strain rate is computed from a viscoplastic potential, exactly as before

$$\dot{\varepsilon}_{ij}^{p} = \frac{\partial g}{\partial \sigma_{ij}}$$

where $g(\sigma_{ij}, T, \text{state variables})$ is a flow potential, and is a function of temperature T and appropriate state variables (yield stress, kinematic variables, etc).

It is common to use a potential based on the Von-Mises yield surface. To do this, define the effective stress and strain rate

$$\sigma_e = \sqrt{\frac{2}{3}S_{ij}S_{ij}} \qquad \dot{\varepsilon}_e = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{p}\dot{\varepsilon}_{ij}^{p}}$$

Then set

$$g = g\left(\sigma_e, T, \text{state variables}\right)$$

The plastic strain rate then follows as

$$\dot{\varepsilon}_{ij}^{p} = \frac{\partial g}{\partial \sigma_e}\frac{3}{2}\frac{S_{ij}}{\sigma_e}$$

The function *g* itself is arbitrary, provided *g* is a nondecreasing function of effective stress. A power-law is particularly popular

$$g = \frac{\dot{\varepsilon}_0}{m+1} \frac{\sigma_e^{\,m+1}}{\sigma_0^{\,m}}$$

where $\dot{\varepsilon}_0$, σ_0 and *m* are (almost certainly temperature dependent) material constants. The strain rate then follows as

$$\dot{\varepsilon}_{ij}^{p} = \frac{3}{2} \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0} \right)^{m} \frac{S_{ij}}{\sigma_e}$$

Note that the rate dependent constitutive law is actually somewhat simpler than the rate independent case, because there is no need to worry about loading-unloading conditions.

One can extend this framework to account for strain hardening (in this case a progressive decrease in strain rate with increasing plastic strain) by making σ_0 increase with strain.

For example, power-law hardening can be included by setting

$$\sigma_0 = Y \left(\frac{\varepsilon_e}{\varepsilon_0} \right)^{1/n}$$

where $\varepsilon_e = \int \dot{\varepsilon}_e dt$ and Y, ε_0 and *n* are material constants.