

INTRODUCTION TO COMPUTATIONAL PLASTICITY

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Chapter 1

Plasticity for absolute beginners

This section intends to provide motivations and basic hypotheses of elasto-plastic theory and presents some of the more classical results of the theory.

In particular, we present the fundamental theorems of limit analysis (the static and kinematic one) and the Melan theorem for plastic adaptation (shake-down analysis).

The discussion clarifies the real meaning of traditional procedures based on elastic analysis and admissible tension criteria.

1.1 Initial remarks

- The structural behaviour is essentially *nonlinear*.
- Concepts like *Failure* or *Structural safety* can not be framed within a linear relationship between causes and effects.
- Traditional analysis procedures, based on elastic solutions and admissible tension criteria, have only a conventional basis.
- The great computational power of new low-cost computers allows, now and even more in the future, a generalized use of more sophisticated procedures, based on nonlinear analysis algorithms.
- The new European norms on buildings (Euro-codes) are based on a nonlinear analysis philosophy.

Therefore, the concepts and procedures of nonlinear analysis will have an increasing importance in the future.

Remarks:

- The nonlinear behaviour of structures descends both from physical aspects (nonlinear constitutive laws: *Plasticity*) and geometrical aspects (nonlinear displacements-deformations laws: *Instability*).
- Only the first theme is treated here.

1.2 Basics of plasticity theory

The elasto–plastic structural behaviour can be framed within a well consolidated theory (*Incremental elasto–plasticity*). The theory is based on the following primitive concepts:

- **Structural materials have limited strength.**

That is, tensions must be limited.

This is formalized assuming that, for each point of the analyzed body, the tension $\sigma := \{\sigma_{xx}, \sigma_{xy}, \dots, \sigma_{zz}\}$ lies within a domain of the tension space, called *elastic domain* of the material

$$\mathcal{D}_e := \{\sigma : f[\sigma] \leq 1\}$$

(*plastic admissibility condition*).

The function $f[\sigma]$ depends, in general, on state parameters taking into account the differences in the behaviour which are related to mechanical or thermal processes (hardening, fatigue, ...).

- **Irreversible deformations are produced.**

This means that in a loading–unloading cycle the structure does not completely recover the initial configuration.

This is formalized by decomposing the deformation increment in two components:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p$$

$\dot{\epsilon}^e$ is the *elastic* part and $\dot{\epsilon}^p$ is the *plastic* part. Only the first one is related to the increment of the tension:

$$\dot{\sigma} = \mathbf{E} \dot{\epsilon}^e$$

1.3 Some more

- **At low tension levels the behaviour is nearly elastic.**

This is formalized by assuming that the plastic component of the incremental deformation, $\dot{\boldsymbol{\epsilon}}^p$, is different from zero only if the tension belongs to the boundary of \mathcal{D}_e (*yield surface*):

$$\dot{\boldsymbol{\epsilon}}^p \neq 0 \text{ only if } f[\boldsymbol{\sigma}] = 1$$

For smaller values of $\boldsymbol{\sigma}$ (smaller in the sense just specified), the incremental behaviour is purely elastic.

Remarks:

- The problem is governed by *equations* (equilibrium, kinematic compatibility, elasticity) and *inequalities* (plastic admissibility).
- The elasto-plastic behaviour is essentially *non-holonomic*. The presence of residual deformations implies that the tension and deformation state due to the application of a load depends not only on the final load attained but also on the loading process (*path-dependence*)
- The constitutive laws are expressed in *incremental form*; this allows an easier treatment of the different behaviours during the loading and unloading phase.

1.4 The Drucker postulate (1951)

The elasto–plastic behaviour can be more precisely described once it is accepted the following postulate, proposed by Drucker:

- *Let the structure be subjected to a loading–unloading cycle due to the application of some additional forces. We have:*

1) during the loading phase, the work done by the additional forces is non-negative;

2) during the entire cycle, the work done by the additional forces is non-negative.

Considering a loading–unloading cycle which takes the tension from an initial admissible state $\boldsymbol{\sigma}_a$ (that is, belonging to \mathcal{D}_e) to a point $\boldsymbol{\sigma}_y$ lying on the yield surface and then returns to $\boldsymbol{\sigma}_a$, the postulate gives:

$$(\boldsymbol{\sigma}_y - \boldsymbol{\sigma}_a)^T \dot{\boldsymbol{\epsilon}}^p \geq 0$$

Since the condition must be true for each $\boldsymbol{\sigma}_a \in \mathcal{D}_e$ and each $\boldsymbol{\sigma}_y \in \partial\mathcal{D}_e$, the postulate leads to the following results:

- *The elastic domain \mathcal{D}_e is convex.*
- *The plastic deformation $\dot{\boldsymbol{\epsilon}}^p$ belongs to the sub-gradient of \mathcal{D}_e , that is to say, it is “normal” to the yield surface.*

1.5 Limit analysis

Using the few results obtained is now possible to give an answer to the following problem:

- *Let the structure be subjected to a proportional loading $\lambda \mathbf{p}$, evaluate the maximum value for the amplifying factor λ .*

The technical relevance of this problem is evident: if \mathbf{p} corresponds to the nominal load, the evaluation of λ_{max} implies an estimate of the failure safety of the structure.

Remarks:

- The problem makes sense only if the function $f[\boldsymbol{\sigma}]$ is constant with the time (*elastic-perfectly plastic behaviour*) or, at least, if it presents a bounded limit surface.
- The presence of isolated plastic zones, embedded in an elastic matrix limiting their deformation, does not represent necessarily a cause of structural danger.
- The situation is different when the plastic deformations, not surrounded by an elastic ring, form a kinematically admissible mechanism.

$$\dot{\boldsymbol{\epsilon}}^p \Longleftarrow \dot{\mathbf{u}}^p$$

- This last occurrence, to which we will refer using the index c , corresponds to the collapse of the structure.

1.6 An example

- The plasticization of the central bar alone, does not cause a plastic mechanism, which is prevented by the lateral bars behaving elastically.

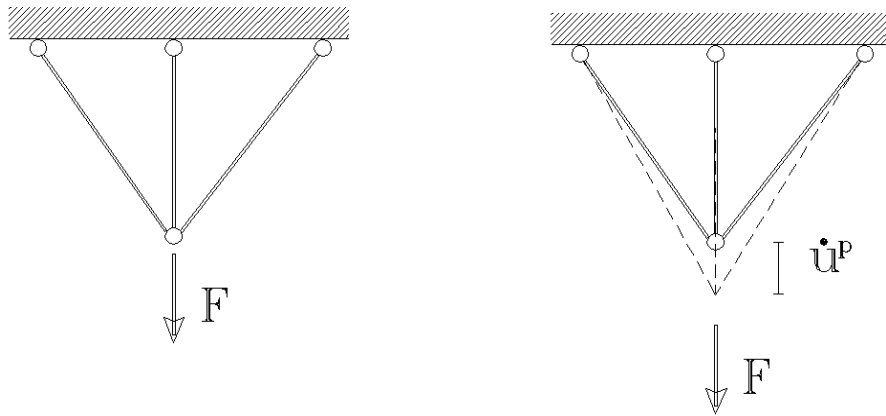


Figure 1.1: 3-bars simple structure

1.7 The static theorem of limit analysis

Consider an elastic–perfectly plastic structure and let

$$\boldsymbol{\sigma}_c, \quad \dot{\boldsymbol{\varepsilon}}^c, \quad \dot{\mathbf{u}}^c, \quad \lambda_c$$

be the tension, the incremental plastic deformation, the incremental plastic displacement ($\dot{\boldsymbol{\varepsilon}}^c$ and $\dot{\mathbf{u}}^c$ are assumed to be kinematically compatible) and the load multiplying factor at collapse. Further, let \mathbf{q} and $\bar{\mathbf{f}}$ be the body and surface external nominal loads.

With regard to a generic *statically admissible* tension field $\boldsymbol{\sigma}_a$ (that is to say, satisfying equilibrium and plastic admissibility conditions), the virtual work equation provides:

$$\begin{aligned} \int_B \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}^c dv &= \lambda_c \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^c dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^c ds \right\} \\ \int_B \boldsymbol{\sigma}_a^T \dot{\boldsymbol{\varepsilon}}^c dv &= \lambda_a \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^c dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^c ds \right\} \end{aligned}$$

where λ_a is the load multiplier associated to $\boldsymbol{\sigma}_a$.

Subtracting these equation, we obtain:

$$\int_B (\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_a)^T \dot{\boldsymbol{\varepsilon}}^c dv = (\lambda_c - \lambda_a) \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^c dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^c ds \right\}$$

The first member is non-negative, and the integral within brackets in the second member is positive (as a consequence of the Drucker postulate, if it is assumed that $\boldsymbol{\sigma}$ is internal to the elastic domain). As a consequence we can derive the following inequality:

$$\lambda_c \geq \lambda_a$$

which corresponds to the **static theorem of limit analysis**):

- *The collapse multiplier is the maximum of all the static multipliers, that is to say, those multipliers associated with statically admissible tension fields.*

1.8 The kinematic theorem of limit analysis

Let $\{\dot{\mathbf{u}}^p, \dot{\boldsymbol{\varepsilon}}^p\}$ be a generic plastic, kinematically admissible mechanism.

In each point of the body for which $\dot{\boldsymbol{\varepsilon}}^p \neq 0$, we can associate the tension $\boldsymbol{\sigma}_p$ to the known plastic deformation, defined by the normality law.

Then, defining the associated kinematical multiplier λ_p through the energy balance:

$$\int_B \boldsymbol{\sigma}_p^T \dot{\boldsymbol{\varepsilon}}^p dv = \lambda_p \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^p dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^p ds \right\}$$

and recalling the equilibrium equation

$$\int_B \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}^p dv = \lambda_c \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^p dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^p ds \right\}$$

we obtain, by difference:

$$\int_B (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_c)^T \dot{\boldsymbol{\varepsilon}}^p dv = (\lambda_p - \lambda_c) \left\{ \int_B \mathbf{q}^T \dot{\mathbf{u}}^p dv + \int_{\partial B} \mathbf{f}^T \dot{\mathbf{u}}^p ds \right\}$$

where the first member is non-negative and the bracketed integrals are positive. As a consequence we derive the inequality:

$$\lambda_p \geq \lambda_c$$

which corresponds to the **kinematic theorem of limit analysis**):

- *The collapse multiplier is the minimum of all kinematic multipliers, that is to say, those multipliers associated with admissible plastic mechanisms.*

1.9 Additional comments

- Both theorems provide neither the failure tension field nor the failure mechanism, but only the collapse multiplier.
- The failure load does not depend on the initial conditions and on the loading process.
- Thanks to the static theorem, the traditional analysis based on nominal elastic solutions and admissible tension criteria, gets a rational meaning: in fact, as a consequence of the use of an equilibrated and plastic admissible tension field, it provides a *limit elastic multiplier* λ_e , actually representing a lower bound for the failure multiplier.
- Possible errors related to the loss of information about the initial tension state (which is generally partially available or not available at all) are irrelevant to this scope.

1.10 Plastic adaptation (shake-down)

- When the structure is subjected to repeated loading–unloading cycles, the plastic failure is not the principal cause of structural danger.
- Even if the failure condition is never attained during the loading process, new plastic deformations can nucleate at each loading cycle.
- for repeated cycles, the total plastic deformation can consequently grow indefinitely or, anyway, (if successive deformations balance each other) can bring to fatigue damage.
- In both cases, the loading process implies the structural ruin.
- Therefore, it is necessary that the plastic process ends rapidly, that is to say, after few cycles (the running-in period) the structure gets again a purely elastic behaviour.
- When this occurs, we say that the structure gets *plastic adaptation*.

1.11 The Melan's theorem (1936)

Let's consider the loading process $\mathbf{p}[t]$ and let

$$\begin{aligned}\boldsymbol{\sigma}^*[t] &= \boldsymbol{\sigma}_E[t] + \Delta\boldsymbol{\sigma}[t] \\ \boldsymbol{\varepsilon}^*[t] &= \boldsymbol{\varepsilon}_E[t] + \Delta\boldsymbol{\varepsilon}[t] + \boldsymbol{\varepsilon}^p[t]\end{aligned}$$

be the out coming elasto-plastic solution, expressed in terms of the elastic solution $\boldsymbol{\sigma}_E[t]$ e $\boldsymbol{\varepsilon}_E[t]$ and of the difference $\Delta\boldsymbol{\sigma}[t]$ (corresponding to an auto-tension field).

We consider (if it does exist) a *nominal* elastic solution, always internal to the elastic domain:

$$\boldsymbol{\sigma}[t] = \boldsymbol{\sigma}_E[t] + \boldsymbol{\sigma}_0, \quad f[\boldsymbol{\sigma}[t]] < 1$$

this solution corresponds to $\boldsymbol{\sigma}_E[t]$, except for the tension $\boldsymbol{\sigma}_0$, which represents the possible difference in the evaluation of the initial stress state (obviously, this difference will be an auto-tension field).

Introducing the quantity:

$$\Psi[t] := \int_B (\boldsymbol{\sigma}^* - \boldsymbol{\sigma})^T \mathbf{E}^{-1} (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}) dv \geq 0$$

we obtain:

$$\dot{\Psi}[t] = - \int_B (\boldsymbol{\sigma}^* - \boldsymbol{\sigma})^T \dot{\boldsymbol{\varepsilon}}^p dv \leq 0 \quad (\dot{\Psi} < 0 \text{ if } \dot{\boldsymbol{\varepsilon}} \neq 0)$$

Hence $\Psi[t]$ is both non-negative (bounded from below) and always decreasing during the plastic process. This implies that the plastic process will surely stop.

As a consequence, we derive the following statement:

- *If a nominal elastic solution (internal to the elastic domain), does exist, the structure will get plastic adaptation.*

1.12 Relations with admissible-tensions criteria

The previous results clarify the real meaning of procedure based on admissible tensions criteria.

- The traditional analysis, based on (nominal) elastic solutions and admissible tensions criteria, provides a lower bound for both the plastic collapse and the plastic shake-down.
- An erroneous evaluation of the initial tension state is irrelevant.
- The use of procedure based on limit analysis concepts alone is not safe with respect to shake-down problems.
- The plastic adaptation theory provides a synthetic approach to the analysis, which does not require a complete information about the time evolution of the loading process, but only requires the knowledge of the maximum value attained by the tensions.
- Anyway, the Melan's theorem does not give any detailed information about the extension that the plastic phase reaches before the structure gets adaptation.
- A complete information can be only provided by a real elasto-plastic analysis based on path-following solution algorithms.

Chapter 2

Finite-step incremental analysis

In this part some theoretical basis of the *incremental elasto-plastic analysis* will be introduced.

In particular it will be presented an analysis procedure oriented to furnish the complete time evolution of the structural response due to a given loading process.

The solution, represented by a path in the load/displacement space, is obtained through the computation of a sequence of equilibrium points $\{\mathbf{u}_k, \mathbf{p}_k\}$ fine enough to reconstruct an interpolating curve.

Since the earlier 60's, the interest for these topics has been strongly increasing, thanks to the availability of powerful computers and to the development of specific numerical algorithms.

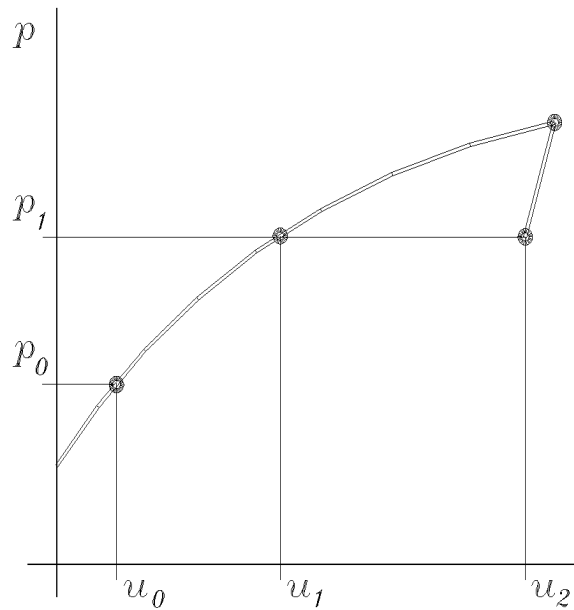
2.1 Holonomic plasticity

In order to obtain incremental elasto–plastic solutions (using small –but necessarily finite– loading steps), it is necessary to solve the following problem:

- Let $\{\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}_0\}$ be a known initial state and $(\mathbf{p}_1 - \mathbf{p}_0)$ a given load increment, determine the corresponding step–end solution $\{\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\}$.

Remarks:

- Because of the irreversibility of the plastic behaviour and the path–dependence of the final state, the set of data which defines the problem is not complete.
- Referring to the figure below, both \mathbf{u}_1 and \mathbf{u}_2 are possible solutions for the load increment $(\mathbf{p}_1 - \mathbf{p}_0)$.



2.2 The Haar–Karman principle (1908)

A possible way for getting an holonomic law in the step is to express directly the equation of the incremental elasto–plastic theory in terms of finite increments. In this way, the step–end solution can be characterized by the following extremal condition (Haar–Karman):

$$\Pi[\boldsymbol{\sigma}] := \frac{1}{2} \int_B \boldsymbol{\sigma}^T E^{-1} \boldsymbol{\sigma} dv + \int_B \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}_0^p dv + \int_{\partial B} (\mathbf{N} \boldsymbol{\sigma})^T \bar{\mathbf{u}} ds$$

under the following constraints:

$$\boldsymbol{\sigma}: \text{ satisfying equilibrium conditions, } f[\boldsymbol{\sigma}] \leq 1$$

In fact, recalling the relations:

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \\ \boldsymbol{\sigma} &= \mathbf{E} \boldsymbol{\varepsilon}^e \end{aligned}$$

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_a)^T \Delta \boldsymbol{\varepsilon}^p \geq 0$$

where $\Delta \boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}_0^p$, we obtain:

$$\begin{aligned} \delta \Pi &= \int_B \delta \boldsymbol{\sigma}^T (\boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}_0^p) dv - \int_B (\mathbf{N} \delta \boldsymbol{\sigma})^T \mathbf{u} ds \\ &= \int_B \delta \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}^T dv - \int_{\partial B} (\mathbf{N} \delta \boldsymbol{\sigma})^T \mathbf{u} ds - \int_B \delta \boldsymbol{\sigma}^T \Delta \boldsymbol{\varepsilon}^p dv \\ &= \int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}_a)^T \Delta \boldsymbol{\varepsilon}^p dv \geq 0 \end{aligned}$$

The Haar–Karman principle can be expressed in the following way:

- *The elasto–plastic solution minimizes the total complementary energy of the system under the equilibrium and plastic admissibility constraints.*

2.3 Reformulation in terms of elastic prediction

Denoting with $\boldsymbol{\sigma}_E$ the “elastic” solution obtained from the initial (step–beginning) state and the assigned load increment, the Haar–Karman principle can be rewritten as:

$$\Pi[\boldsymbol{\sigma}] := \frac{1}{2} \int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}_E)^T \mathbf{E}^{-1} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_E) dv = \text{minimum}$$

under the constraints:

$$\boldsymbol{\sigma}: \text{ satisfying equilibrium conditions, } f[\boldsymbol{\sigma}] \leq 1$$

It is worth observing that the functional $\Pi[\boldsymbol{\sigma}]$ represents, in an energy norm, the square of the distance between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_E$. Hence the principle can be stated as:

- *Among the stress states verifying equilibrium and plastic admissibility conditions, the step–end elasto–plastic solution $\boldsymbol{\sigma}$ has the minimum distance (in an energy norm) from the elastic solution $\boldsymbol{\sigma}_E$ of the same problem.*

Remarks:

- This formulation is particularly convenient from both the numerical and the computational point of view.
- The solution can be easily characterized. If the point $\boldsymbol{\sigma}$ belongs to the elastic domain, we have $\boldsymbol{\sigma} = \boldsymbol{\sigma}_E$. Otherwise, $\boldsymbol{\sigma}$ is found as the tangential contact point between two convex surfaces: the yield surface and a level line of the elastic strain energy.
- Since the latter surface is strictly convex, the uniqueness of the solution is proved.

2.4 The Ponter–Martin extremal paths theory

The extremal paths theory, formulated by Ponter & Martin in 1972, furnishes a convenient link between incremental and holonomic plasticity. The following results hold:

- Among all the incremental elasto–plastic paths starting from the initial state $\{\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}_0\}$, there are some *extremal paths* that realize both the minimum of deformation work (for fixed step–end strain $\boldsymbol{\varepsilon}_1$) and the maximum complementary work (for fixed step–end stress $\boldsymbol{\sigma}_1$)
- The use of extremal paths defines a holonomic law in the step that, for Drucker’s materials, satisfies the conditions:

$$0 \leq (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) \leq (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)^T \mathbf{E} (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)$$

- For elastic perfectly–plastic materials, the extremal path solution corresponds to the one defined by the Haar–Karman principle.

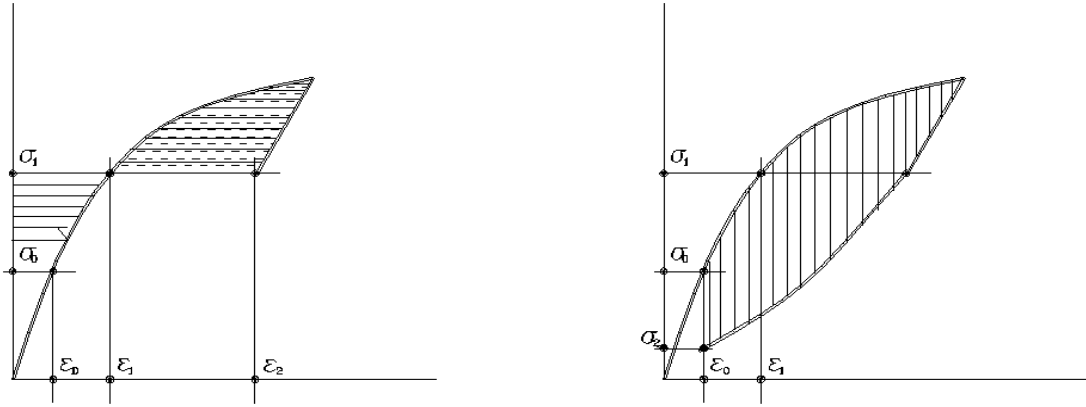


Figure 2.1: Extremal solution $(\{\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1\})$

2.5 A deep insight

Let $\boldsymbol{\sigma}[t]$ be a path between $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ in the stress space and let $\boldsymbol{\varepsilon}[t]$ be the corresponding path in the strain space, image of $\boldsymbol{\sigma}[t]$ through the constitutive law.

The complementary work on $\boldsymbol{\sigma}[t]$ is defined as

$$U[\boldsymbol{\sigma}[t]] := \int_{\boldsymbol{\sigma}[t]} \dot{\boldsymbol{\sigma}}^T \boldsymbol{\varepsilon} dt$$

We define *extremal* the paths characterized by the following (extremal) condition:

$$\hat{U}[\boldsymbol{\sigma}_1] := U[\hat{\boldsymbol{\sigma}}[t]] \geq U[\boldsymbol{\sigma}[t]]$$

In order to complete the definition domain of \hat{U} , we assume that

$$\hat{U}[\boldsymbol{\sigma}_1] := +\infty$$

when no admissible path between $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ does exist. So, $\hat{U}[\boldsymbol{\sigma}_1]$ has been characterized as a function defined in all the $\boldsymbol{\sigma}$ space, and will be called *extremal-path potential*.

From the decomposition of the total strain in the elastic and plastic components ($\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$), it is possible to derive the decomposition of the complementary work in the elastic part $U^e[\boldsymbol{\sigma}]$ and the plastic part $U^p[\boldsymbol{\sigma}]$. Only the latter is path-dependent, and we can express it in this way:

$$U^p[\boldsymbol{\sigma}[t]] := \int_{\boldsymbol{\sigma}[t]} \dot{\boldsymbol{\sigma}}^T \boldsymbol{\varepsilon}^p dt$$

Then, by difference, we obtain the *extremal-path plastic potential*

$$\hat{U}^p[\boldsymbol{\sigma}_1] = \hat{U}[\boldsymbol{\sigma}_1] - U^e[\boldsymbol{\sigma}_1]$$

2.6 Convexity of the extremal-path potential

Let's consider an extremal path between $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$, and a different path made up of an extremal part between $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_2$ and a linear part between $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_1$:

$$\boldsymbol{\sigma}_L[t] = \boldsymbol{\sigma}_2 + t\dot{\boldsymbol{\sigma}}_L, \quad \dot{\boldsymbol{\sigma}}_L = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$$

We have, as a consequence of the previous definition:

$$\begin{aligned} \hat{U}[\boldsymbol{\sigma}_1] &\geq \hat{U}[\boldsymbol{\sigma}_2] + \int_0^1 (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T (\hat{\boldsymbol{\varepsilon}}_2 + \Delta\boldsymbol{\varepsilon}_L[t]) dt \\ &= \hat{U}[\boldsymbol{\sigma}_2] + (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \hat{\boldsymbol{\varepsilon}}_2 + \int_0^1 (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \Delta\boldsymbol{\varepsilon}_L[t] dt \end{aligned}$$

The last expression can be rewritten

$$\int_0^1 (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^T \Delta\boldsymbol{\varepsilon}_L[t] dt = \int_0^1 \left\{ \int_0^t \dot{\boldsymbol{\sigma}}_L^T \dot{\boldsymbol{\varepsilon}}_L d\tau \right\} dt$$

For Drucker's materials (which satisfy the condition $\dot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\varepsilon}} \geq 0$) this expression is non-negative.

We can derive, therefore, the following fundamental inequality:

$$\hat{U}[\boldsymbol{\sigma}_1] - \hat{U}[\boldsymbol{\sigma}_2] - \hat{\boldsymbol{\varepsilon}}_2^T (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \geq 0$$

As a consequence, using the standard results of convex analysis, we obtain some important results:

1. The functional $\hat{U}[\boldsymbol{\sigma}_1]$ is convex.
2. The strain $\hat{\boldsymbol{\varepsilon}}$ belongs to the sub-differential $\partial\hat{U}$ of \hat{U} :

$$\hat{\boldsymbol{\varepsilon}}[\boldsymbol{\sigma}_1] \in \partial\hat{U}[\boldsymbol{\sigma}_1] := \{ \boldsymbol{\eta} : \hat{U}[\boldsymbol{\sigma}_2] - \hat{U}[\boldsymbol{\sigma}_1] - \boldsymbol{\eta}^T (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) \geq 0, \forall \boldsymbol{\sigma}_2 \}$$

In the same way it is possible to prove the convexity of the extremal-path plastic potential $\hat{U}^p[\boldsymbol{\sigma}_1]$ and the normality law for the plastic strain $\hat{\boldsymbol{\varepsilon}}^p$

$$\hat{\boldsymbol{\varepsilon}}^p[\boldsymbol{\sigma}_1] \in \partial\hat{U}^p[\boldsymbol{\sigma}_1]$$

2.7 Consequences of convexity

Let $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ be two step-end stresses; using the property of convexity of the extremal potential, we obtain

$$\hat{U}[\boldsymbol{\sigma}_2] - \hat{U}[\boldsymbol{\sigma}_1] - \hat{\boldsymbol{\varepsilon}}_1^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) \geq 0$$

$$\hat{U}[\boldsymbol{\sigma}_1] - \hat{U}[\boldsymbol{\sigma}_2] - \hat{\boldsymbol{\varepsilon}}_2^T(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \geq 0$$

then, adding up the two members of the equations:

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1) \geq 0$$

In the same way, using the property of convexity of the extremal-path plastic potential, we obtain:

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^T(\hat{\boldsymbol{\varepsilon}}_2^p - \hat{\boldsymbol{\varepsilon}}_1^p) \geq 0$$

This relation, applying the strain decomposition law:

$$\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^e + \hat{\boldsymbol{\varepsilon}}^p, \quad \boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}^e$$

which implies that

$$\begin{aligned} (\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T \mathbf{E}(\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1) &= (\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) \\ &\quad + (\hat{\boldsymbol{\varepsilon}}_2^p - \hat{\boldsymbol{\varepsilon}}_1^p)^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) + (\hat{\boldsymbol{\varepsilon}}_2^p - \hat{\boldsymbol{\varepsilon}}_1^p)^T \mathbf{E}(\hat{\boldsymbol{\varepsilon}}_2^p - \hat{\boldsymbol{\varepsilon}}_1^p), \end{aligned}$$

provides the condition:

$$(\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T \mathbf{E}(\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1) \geq (\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)$$

The two previous inequalities can be rearranged in the following form:

$$0 \leq (\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) \leq (\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)^T \mathbf{E}(\hat{\boldsymbol{\varepsilon}}_2 - \hat{\boldsymbol{\varepsilon}}_1)$$

Later on, this inequality will have a great importance.

2.8 Principle of Minimum for Drucker's materials

Let $\boldsymbol{\sigma}$ be the elasto-plastic extremal-path end-step solution, and $\boldsymbol{\sigma}_{eq}$ a generic stress field, in equilibrium under the same end-step loads. The following extremal condition must hold:

$$\hat{U}[\boldsymbol{\sigma}_{eq}] - \hat{U}[\boldsymbol{\sigma}] - \hat{\boldsymbol{\varepsilon}}[\boldsymbol{\sigma}]^T(\boldsymbol{\sigma}_{eq} - \boldsymbol{\sigma}) \geq 0$$

Moreover, since $\boldsymbol{\sigma}_{eq} - \boldsymbol{\sigma}$ is an auto-tension field, we have:

$$\int_B \hat{\boldsymbol{\varepsilon}}[\boldsymbol{\sigma}]^T(\boldsymbol{\sigma}_{eq} - \boldsymbol{\sigma}) dv - \int_{\partial B} (\mathbf{N}(\boldsymbol{\sigma}_{eq} - \boldsymbol{\sigma}))^T \mathbf{u} ds = 0$$

Therefore, by integrating the inequality over the domain B , we obtain:

$$\int_B \hat{U}[\boldsymbol{\sigma}] dv - \int_{\partial B} (\mathbf{N}\boldsymbol{\sigma})^T \mathbf{u} ds \leq \int_B \hat{U}[\boldsymbol{\sigma}_{eq}] dv - \int_{\partial B} (\mathbf{N}\boldsymbol{\sigma}_{eq})^T \mathbf{u} ds$$

This result can be expressed with the following statement:

- *Among all the equilibrated stress fields, the elasto-plastic extremal-path solution minimizes the extremal-path potential.*

The statment represents a generalization of the principle of minimum of the total complementary energy, valid for elasto-plastic, stable materials which satisfy the normality law

$$\hat{\boldsymbol{\varepsilon}}^p[\boldsymbol{\sigma}] \in \partial \hat{U}^p[\boldsymbol{\sigma}]$$

2.9 Principle of Minimum for elastic perfectly-plastic materials

With respect to elastic perfectly-plastic materials, the following conditions hold:

- any admissible stress $\boldsymbol{\sigma}$ can be reached through a purely elastic path.
- let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_a$ be two admissible stresses and $\dot{\boldsymbol{\epsilon}}^p$ the incremental plastic strain associated to $\boldsymbol{\sigma}$, we have:

$$(\boldsymbol{\sigma}_a - \boldsymbol{\sigma})^T \dot{\boldsymbol{\epsilon}} \leq 0$$

Moreover, the path-depended part of the complementary work can be written as:

$$\begin{aligned} \int_0^T (\boldsymbol{\epsilon}^p[t] - \boldsymbol{\epsilon}^p[0])^T \dot{\boldsymbol{\sigma}}[t] dt &= \int_0^T \left\{ \int_0^t \dot{\boldsymbol{\epsilon}}^p[\tau]^T d\tau \right\} \dot{\boldsymbol{\sigma}}[t] dt \\ &= \int_0^T \dot{\boldsymbol{\epsilon}}^p[t]^T \left\{ \int_t^T \dot{\boldsymbol{\sigma}}[\tau] d\tau \right\} dt = \int_0^T \dot{\boldsymbol{\epsilon}}^p[t]^T (\boldsymbol{\sigma}_a - \boldsymbol{\sigma}[t]) dt \leq 0 \end{aligned}$$

and this proves that the complementary work is maximized when $\dot{\boldsymbol{\epsilon}}^p = 0$.

As a consequence, the extremal paths can be characterized as purely elastic paths. Then we can write:

$$\hat{U}[\boldsymbol{\sigma}] = \begin{cases} U^e & \text{se } f[\boldsymbol{\sigma}] \leq 1 \\ +\infty & \text{se } f[\boldsymbol{\sigma}] > 1 \end{cases}$$

and the principle of minimum becomes equivalent to the Haar-Karman principle:

- *Among all the equilibrated and admissible stress fields, the elasto-plastic extremal-path solution minimizes the complementary energy.*

Chapter 3

Computational strategies

This part intends to discuss some numerical strategies suitable for the finite element elasto-plastic analysis of complex structures.

In particular, the discussion will refer to the incremental elasto-plastic problem, using the so called *initial stress* strategy combined with a *path-following incremental method*.

3.1 Numerical algorithms for plastic analysis

1. Limit analysis:

- Linear programming algorithms (based on limit analysis theorems and on the static–kinematic duality).
- Nonlinear programming (algorithms based on limit analysis theorems).
- Alternative variational formulations combined with the use of specialized solution algorithms.

This class of algorithms has been very popular (at least within the Italian academy) during the 60's – 70's. At the present time, it is not used at all, except for rare cases, because it is considered much less efficient than the incremental approach.

2. Plastic adaptation (shake–down):

- The situation is the same as limit analysis.

In a certain sense, the problem is similar to that of limit analysis, and therefore similar solution algorithms could be used. However, the methods recently proposed are poorly efficient, even if there is the possibility of providing synthetic results for complex loading combinations, circumstance that makes this approach particularly interesting.

Because of the technical relevance of the topic, it is desirable an increasing development in the research.

3. Incremental analysis:

- Heuristic algorithms (Euler extrapolation ...).
- Quadratic programming.
- *Initial stress* approach:
 - Secant matrix iteration
 - Newton–Raphson method
 - Modified Newton–Raphson
 - Riks’ arc-length method
 -
- Explicit algorithms based on pseudo–dynamic approach.

The algorithms belonging to this class are widely used, thanks to their good efficiency.

Some implementations of the initial stress method, combined with an incremental Riks strategy, are standard features of the commercial FEM codes.

Pseudo–dynamical algorithms usually require very high computational power.

3.2 Incremental elasto–plastic analysis

Let the structure be modeled by finite elements and let $p[\lambda]$ be the assigned loading path; we have to solve the following problem:

- *Determine a sequence of equilibrium points (u_k, λ_k) fine enough to obtain the path by interpolation.*

The problem has a recurrent structure and can be decomposed in two sub-problems:

- P1: Knowing the initial state (at the beginning of the step) and the final nodal displacement vector \mathbf{u} (at the end of the step), determine the corresponding vector of internal nodal forces, $\mathbf{s}[\mathbf{u}]$.
- P2: At the end of the step the vector of nodal loads \mathbf{p} is given; determine \mathbf{u} such that the following equilibrium equation is satisfied:

$$\mathbf{s}[\mathbf{u}] = \mathbf{p}$$

Remarks:

- Only the subproblem P1 requires the description of the elasto–plastic response of the structure (i.e., contains the physics of the problem).
- The subproblem P2 actually corresponds to the solution of a implicit non–linear vectorial equation.

3.3 Solution of subproblem P1

The theory of extremal paths provides a good theoretical frame for the subproblem P1.

Making use of extremal paths, in fact, the load step is performed through a real incremental elasto–plastic path, which—in addition—presents some theoretical advantages (as shown by Ponter–Martin, a little shift in the trajectory leads to negligible differences in the attained final state).

By integrating the conditions written for the material on the overall structure, we obtain for $\mathbf{s}[\mathbf{u}]$ a representation characterized by the following inequalities:

$$0 \leq (\mathbf{s}[\mathbf{u}_2] - \mathbf{s}[\mathbf{u}_1])^T (\mathbf{u}_2 - \mathbf{u}_1) \leq (\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{K}_E (\mathbf{u}_2 - \mathbf{u}_1)$$

where \mathbf{u}_1 and \mathbf{u}_2 are two possible final displacement vectors and \mathbf{K}_E is the standard elastic stiffness matrix of the structure.

Practically, the nodal forces $\mathbf{s}[\mathbf{u}]$ can be obtained through a standard element–by–element assemblage of the single element contributions. That is to say:

1. For each element we determine the elastic solution due to the initial stress state $\boldsymbol{\sigma}_0$ and to the assigned displacement increment $(\mathbf{u} - \mathbf{u}_0)$ (elastic prediction):

$$\boldsymbol{\sigma}_E = \boldsymbol{\sigma}_0 + \mathbf{E}\boldsymbol{\varepsilon} , \quad \boldsymbol{\varepsilon} := \mathbf{D}(\mathbf{u} - \mathbf{u}_0)$$

2. The nodal displacements in the element are assigned. Therefore there is no nodal equilibrium condition to be satisfied, and no interaction among elements. Consequently, the Haar–Karman minimization condition only implies local, independent conditions defined for each element.

3.4 Example 1: constant tension plane elements

Let $\bar{\sigma}$ and $\tilde{\sigma}_{ij}$ be the cubic and deviatoric components of the final stress, and let $\bar{\sigma}_E$, $\tilde{\sigma}_{Eij}$ be the corresponding elastic predictions.

- The elastic strain energy $\Phi[\boldsymbol{\sigma}]$ and the plastic admissibility condition (Mises) $f[\boldsymbol{\sigma}] \leq 1$ are expressed by:

$$\Phi[\boldsymbol{\sigma}] := \frac{1}{2E} \left\{ (1 + \nu) \sum_{ij} \tilde{\sigma}_{ij}^2 + 3(1 - 2\nu) \bar{\sigma}^2 \right\}$$

$$f[\boldsymbol{\sigma}] := \frac{1}{\sigma_y^2} \sum_{ij} \tilde{\sigma}_{ij}^2 < 1$$

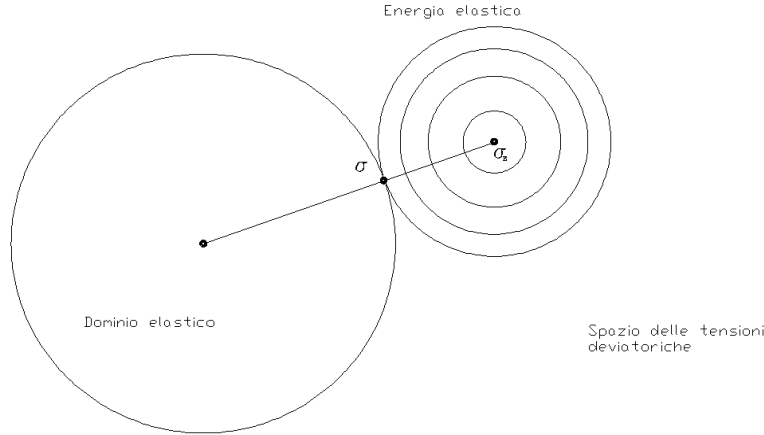


Figure 3.1: Haar-Karman solution

- In the deviatoric space, the level curves of $\Phi[\boldsymbol{\sigma}]$ and $f[\boldsymbol{\sigma}]$ have both spherical shape. The Haar–Karman solution

$$\Phi[\boldsymbol{\sigma}] = \min. , \quad f[\boldsymbol{\sigma}] \leq 1$$

which corresponds to the tangent point, is then obtained as:

$$\tilde{\sigma}_{ij} := \frac{\tilde{\sigma}_{Eij}}{\max(f[\boldsymbol{\sigma}_E]^{1/2}, 1)} , \quad \bar{\sigma} := \bar{\sigma}_E$$

3.5 Some comments

- The solution process described can be employed, more generally, in the case of compatible elements based on numerical integration; in this case it will be applied to each Gauss point.
- In presence of a discontinuity between elastic and plastic zones, it isn't possible to localize with accuracy the interface between the different behaviours in the element. Then, the discretization error is always proportional to the dimension h of the element, even using complex elements, that in elasticity present a better error estimate (proportional to h^n , being $n \approx 2-4$).
- The error usually appears in the form of locking and can be relevant.
- The use of mixed elements is a way to avoid locking phenomena.
- Because of the particular structure of the Haar–Karman principle, hybrid elements, defined by compatible boundary displacements and equilibrated internal stress fields, are particularly convenient.

3.6 Example 2: framed structures

We use a hybrid beam element, defined by the end-sections displacements and rotations and by internally equilibrated fields for the bending moment and the axial strength. The element behaviour depends on 6 kinematic and 3 static variables.

Let N be the axial strength, M_i and M_j the bending moments (in the two edge sections of the element), and N_E , M_{Ei} , M_{Ej} the elastic predictions corresponding to the assigned nodal displacements.

- The complementary strain energy is defined by:

$$\Phi[\boldsymbol{\sigma}] := \frac{1}{2} \left[\frac{\ell}{EA} N^2 + \frac{3\ell}{EJ} (M_i^2 + M_j^2 - 2cM_iM_j) \right]$$

where

$$c := \frac{1 - \beta/2}{2 + \beta} \quad \beta := \frac{12EJ}{GA\ell^2}$$

- The plastic admissibility condition (expressed only for the end sections) provides:

$$-M_y \leq M_i \leq M_y, \quad -M_y \leq M_j \leq M_y$$

- The Haar-Karman solutions is directly obtained from the following 3-steps algorithm:

- 1) $M_i := \max\{-M_y, \min\{M_{Ei}, M_y\}\}$
- 2) $M_j := \max\{-M_y, \min\{M_{Ej} - c(M_{Ei} - M_i), M_y\}\}$
- 3) $M_i := \max\{-M_y, \min\{M_{Ei} - c(M_{Ej} - M_j), M_y\}\}$

3.7 Solution of subproblem P2

The equilibrium at the end of the step is expressed by the condition:

$$\mathbf{s}[\mathbf{u}] = \mathbf{p}$$

which corresponds to a nonlinear implicit equation in the unknown \mathbf{u} .

The equation can be iteratively solved using the *modified Newton–Raphson* method:

$$\begin{aligned}\mathbf{r}_j &:= \mathbf{p} - \mathbf{s}[\mathbf{u}] \\ \mathbf{u}_{j+1} &:= \mathbf{u}_j + \tilde{\mathbf{K}}^{-1} \mathbf{r}_j\end{aligned}$$

where $\tilde{\mathbf{K}}$ is a suitable approximation for the Hessian matrix

$$\mathbf{K}_t[\mathbf{u}] := \frac{d\mathbf{s}[\mathbf{u}]}{d\mathbf{u}}$$

The convergence of the MNR iteration scheme can be discussed introducing the *secant matrix* \mathbf{K}_j , defined in the j -th iteration step by the following equivalence

$$\mathbf{K}_j(\mathbf{u}_{j+1} - \mathbf{u}_j) = \mathbf{s}[\mathbf{u}_{j+1}] - \mathbf{s}[\mathbf{u}_j]$$

The iteration scheme implies

$$\mathbf{r}_{j+1} = [\mathbf{I} - \mathbf{K}_j \tilde{\mathbf{K}}^{-1}] \mathbf{r}_j$$

Therefore, we derive the following (sufficient) convergence condition:

$$\rho[\mathbf{I} - \mathbf{K}_j \tilde{\mathbf{K}}^{-1}] < 1 \quad , \quad \forall j$$

where $\rho[\cdot]$ is the spectral radius of the matrix.

The convergence condition can be rearranged in a more useful form:

$$\mathbf{0} < \mathbf{K}_j < 2\tilde{\mathbf{K}}$$

3.8 Some remarks

- The inequality

$$0 \leq (\mathbf{s}[\mathbf{u}_{j+1}] - \mathbf{s}[\mathbf{u}_j])^T (\mathbf{u}_{j+1} - \mathbf{u}_j) \leq (\mathbf{u}_{j+1} - \mathbf{u}_j)^T \mathbf{K}_E (\mathbf{u}_{j+1} - \mathbf{u}_j)$$

which is true for solutions obtained through extremal paths, starting from \mathbf{u}_0 , can be written in the following matrix form:

$$\mathbf{0} \leq \mathbf{K}_j \leq \mathbf{K}_E$$

Hence, the second part of the convergence condition (i.e., $\mathbf{K}_j < 2\tilde{\mathbf{K}}$) is trivially satisfied once it is assumed (as in the initial stress method)

$$\tilde{\mathbf{K}} := \mathbf{K}_E$$

(or a reasonable approximation of \mathbf{K}_E).

- the first part of the convergence condition ($\mathbf{K}_j > 0$) is more critical because, even if we have $\mathbf{K}_j \geq 0$, the iteration scheme fails to converge near to the collapse, where:

$$\mathbf{K}_t[\mathbf{u}] \rightarrow \mathbf{0}$$

- For this reason, the MNR-based incremental process is not able to describe the collapse state of the structure.
- Actually, when the plasticization process goes on, the convergence of the iterative process rapidly decreases, and this generally means that the process itself will prematurely stop.

3.9 The arc-length method

- The convergence problems arising near the limit point of the equilibrium path are related to the parametric representation adopted to describe this curve. In fact, the representation has been assumed in the form $\mathbf{u} = \mathbf{u}[\lambda]$, even if the curve we are looking for is not analytical in λ .

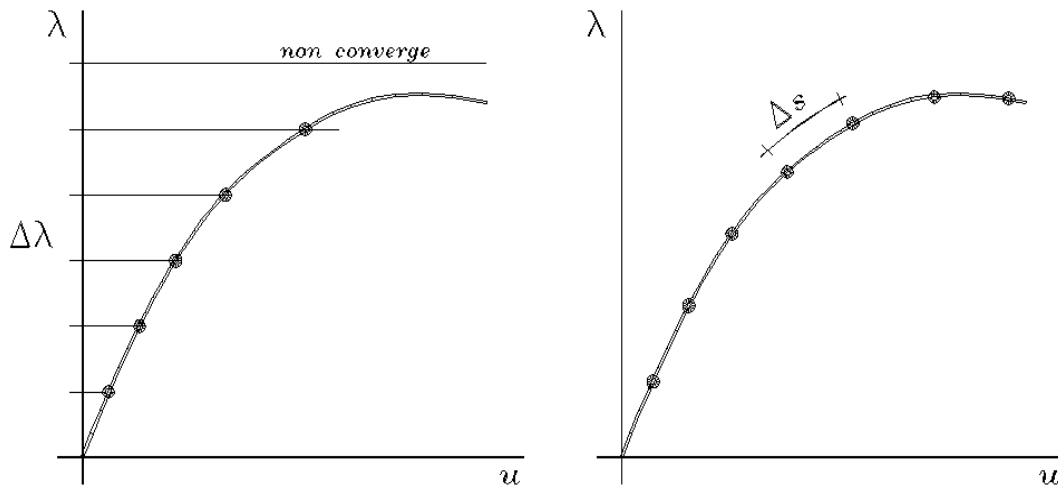


Figure 3.2: Load-controlled analysis and arc-length method

- These difficulties are easily avoided if an analytical representation is used. A proper description is obtained assuming a curvilinear *arc-length* abscissa as description parameter, in the $\{\mathbf{u}, \lambda\}$ space.
- The arc-length method has been proposed by Riks in 1974 for the path-following analysis of nonlinear elastic problems. At the present time, it can be considered as the “de facto” standard method for nonlinear analysis.

3.10 Riks' iteration scheme

The iteration scheme due to Riks represents the first, and yet more efficient, implementation of the arc-length method.

Its main idea is to introduce explicitly the load multiplier λ as further unknown. At the same time a further constraint is needed, and it is given in the following way:

$$\Delta \mathbf{u}^T \mathbf{M} \dot{\mathbf{u}} + \mu \Delta \lambda \dot{\lambda} = 0$$

where \mathbf{M} and μ are appropriate metric factors. This condition implies the orthogonality (in the enlarged space $\{\mathbf{u}, \lambda\}$) between the iterative correction

$$\begin{cases} \dot{\mathbf{u}} &= \mathbf{u}_{j+1} - \mathbf{u}_j \\ \dot{\lambda} &= \lambda_{j+1} - \lambda_j \end{cases}$$

and the total step increment

$$\begin{cases} \Delta \mathbf{u} &= \mathbf{u}_j - \mathbf{u}_0 \\ \Delta \lambda &= \lambda_j - \lambda_0 \end{cases}$$

If the iteration starts from a suitable extrapolation $\{\mathbf{u}_1, \lambda_1\}$ that realizes the required distance from $\{\mathbf{u}_0, \lambda_0\}$, the constraint represents an approximate, but computationally efficient way to fix the arc-length.

With this choice, the iteration scheme can be rearranged:

$$\begin{cases} \tilde{\mathbf{K}} \dot{\mathbf{u}} & - \hat{\mathbf{p}} \dot{\lambda} &= \mathbf{r}_j \\ \Delta \mathbf{u}^T \mathbf{M} \dot{\mathbf{u}} & + \mu \Delta \lambda \dot{\lambda} &= 0 \end{cases}$$

where

$$\hat{\mathbf{p}} := \frac{d\mathbf{p}[\lambda]}{d\lambda}$$

3.11 Convergence of Riks' scheme

The use of the Riks' iteration scheme leads to the relation:

$$\mathbf{r}_{j+1} = [\mathbf{I} - \mathbf{K}_j \tilde{\mathbf{K}}^{-1}] [\mathbf{I} - \alpha_j \mathbf{B}_j] \mathbf{r}_j$$

the matrix \mathbf{B}_j and the factor α_j are defined by:

$$\mathbf{B}_j := \frac{\hat{\mathbf{p}} \mathbf{d}_j^T}{\hat{\mathbf{p}}^T \mathbf{d}_j} \quad , \quad \alpha_j := \frac{\hat{\mathbf{p}}^T \mathbf{d}_j}{\mu \Delta \lambda_j + \hat{\mathbf{p}}^T \mathbf{d}_j}$$

Where we have called:

$$\mathbf{d}_j := \tilde{\mathbf{K}}^{-1} \mathbf{M} \Delta \mathbf{u}_j$$

The following sufficient condition for convergence is derived:

$$\rho \left[[\mathbf{I} - \mathbf{K}_j \tilde{\mathbf{K}}^{-1}] [\mathbf{I} - \alpha_j \mathbf{B}_j] \right] < 1 \quad , \quad \forall j$$

Remarks:

- The better performance of the Riks' scheme is strictly related to the filter effect produced by the matrix

$$[\mathbf{I} - \alpha_j \mathbf{B}_j]$$

- The filter does not affect the components of \mathbf{r}_j orthogonal to \mathbf{d}_j , and reduces the parallel components of a $(1 - \alpha_j)$ factor.
- For $\mu \approx 0$ and, in any case, near to a limit point (where $\Delta \lambda \rightarrow 0$), we have $\alpha_j \approx 0$ and then the filter corresponds to an orthogonal projection on the direction \mathbf{d}_j .

3.12 Implementation of the Riks' scheme

The Riks' scheme is a very powerful tool, but its efficiency partially depends on the appropriate choice of the metric factors μ ed \mathbf{M} .

A convenient choice can be to assume $\mu = 0$ and \mathbf{M} such that

$$\mathbf{d}_j = \hat{\mathbf{u}} := \tilde{\mathbf{K}}^{-1} \hat{\mathbf{p}}$$

(These choice is equivalent to the assumption $\mathbf{M} \approx \mathbf{K}_j$.)

Now, the scheme can be rearranged in the following, explicit form:

$$\begin{cases} \lambda_{j+1} = \lambda_j - \mathbf{r}_j^T \hat{\mathbf{u}} / \hat{\mathbf{p}}^T \hat{\mathbf{u}} \\ \mathbf{u}_{j+1} = \mathbf{u}_j + \tilde{\mathbf{K}}^{-1} \mathbf{r}_j + (\lambda_{j+1} - \lambda_j) \hat{\mathbf{u}} \end{cases}$$

Using arguments similar to the load-controlled scheme, we can derive the following (sufficient) convergence condition:

$$\mathbf{0} < \mathbf{K}_t[\mathbf{u}] < 2\tilde{\mathbf{K}} ; \quad \text{in } \bar{\mathcal{U}} := \{ \dot{\mathbf{u}} : \hat{\mathbf{p}}^T \dot{\mathbf{u}} = 0 \}$$

analogous to the condition obtained in the previous case. Now the inequalities do not need to be satisfied in the whole incremental displacement space, but only in a subspace, orthogonal to the vector $\hat{\mathbf{p}}$.

We have, trivially:

$$\hat{\mathbf{p}}^T \dot{\mathbf{u}}_c = \int_B \sigma_c^T \dot{\varepsilon}_c dv > 0$$

for each non-null collapse mechanism $\dot{\mathbf{u}}_c$. Then, the direction of singularity of the operator $\mathbf{K}_t[\mathbf{u}]$ does not lie in the orthogonal subspace and the global convergence of the scheme is assured.

3.13 Adaptive solution strategy

An efficient incremental process should have an adaptive behaviour; that is, it should be able, on the basis of autonomous choices, to vary its internal parameters in order to reduce the computational cost and increase the accuracy of the analysis.

In particular, the following features are required:

- the process should tune the step length: that is, it should increase such length where the equilibrium curve is weakly nonlinear, and reduce it in the strongly nonlinear parts. This allows either a better description of the curve and the computation of a smaller number of equilibrium points.
- The process should tune the iteration matrix $\tilde{\mathbf{K}}$ too, in order to follow the evolution of \mathbf{K}_j : for instance, we should use the full elastic stiffness matrix only in the nearly elastic incremental steps, and a reduced stiffness in the plastic steps.
- The adaptivity feature must be computationally cheap and, anyway, must preserve the reliability of the overall analysis.

3.14 Some implementation details

It is convenient to introduce two adaptive parameters: β and ω .

- The first one controls the initial extrapolation

$$\begin{cases} \mathbf{u}_1 = \mathbf{u}_0 + \beta \Delta \mathbf{u}_0 \\ \lambda_1 = \lambda_0 + \beta \Delta \lambda_0 \end{cases}$$

where $\Delta \mathbf{u}_0$ and $\Delta \lambda_0$ are the total increments attained in the previous step.

- The second one controls the evaluation of the iteration matrix $\tilde{\mathbf{K}}$, which is assumed to be proportional to \mathbf{K}_E

$$\tilde{\mathbf{K}} = \frac{1}{\omega} \mathbf{K}_E$$

(The use of this scalar does not bring additional computational costs: in fact, being $\tilde{\mathbf{K}}^{-1} = \omega \mathbf{K}_E^{-1}$, it is possible to use the matrix \mathbf{K}_E which is assembled and decomposed once for all)

The following formulas can be used:

- in the j -th iteration loop,

$$\omega_{j+1} = \omega_j \frac{\mathbf{r}_j^T \dot{\mathbf{u}}_j}{(\mathbf{r}_j - \mathbf{r}_{j+1})^T \dot{\mathbf{u}}_j} \quad \text{with the limits } 0 < \omega < 2$$

- in the k -th step of the incremental process,

$$\beta_{k+1} = \beta_k t^{\frac{n-\bar{n}}{2n\bar{n}}}$$

where n is the number of iteration loops performed in the previous step, \bar{n} is the number of desired loops (usually 4–10) and t is the relative tolerance required for the equilibrium.

Chapter 4

Finite element discretization

The presence of the discontinuities related to the elastoplastic behaviour requires an approach different from the one generally used for elastic problems, for which the solution is characterized by an high degree of regularity.

Discretization methods (finite element modeling) which are efficient for elastic problems can be not so convenient for plastic ones.

In this part a rapid sketch on this topics is given, with the aim to describe some finite elements potentially suitable for elasto-plastic problems.

4.1 Some initial remarks

- Within plastic zones, because of the discontinuities in the strain field (introduced by the plastic behaviour), the finite element discretization error depends linearly on the size h of the mesh, even using high interpolation elements.
- Therefore the use of complex elements, characterized by a high number of variables per element, is not advantageous.
- The accuracy must be attained through the mesh refinement.
- All this consideration suggests the use of simple elements, with few variables per element, and fine meshes.
- An accurate evaluation of the stress field is more important than in elasticity, because the material behaviour is strictly related to the level of stress.
- From this point of view, compatible elements, which evaluate displacements more accurately than stresses (the latter are only obtained by derivation from the former), could represent a bad choice.
- Mixed or Hybrid elements, for which the accuracy for displacements and stresses can be balanced, seem to be more convenient.

4.2 Further comments

- Since plastic deformations are typically localized on slip (yield) planes, above all when the load approaches the collapse level, it is convenient the use of elements suitable to allow discontinuity both in the tension and in the strain field.
- With respect to kinematics, elements should assure the continuity, on contact surfaces, of normal displacements but not of tangential ones (just the opposite of what usually happens).
- With respect to statics, elements should assure the continuity of normal and tangential stresses but not of transversal ones (just the opposite of what usually happens).
- A compromise between different continuity requirements is not easy and does not lead to a simple algebraic formulation of the element. Moreover, the use of discontinuous fields could be inappropriate for the part of the body which remains elastic.
- Apparently, the only alternative is the use of very fine meshes, such that discontinuities can develop in a band of small thickness (comparable to the mesh size).

4.3 Simplex mixed elements

- These elements are triangular (more generally, a tetrahedron with $n + 1$ vertices, n being the space dimension of the problem).
- Both displacement and stress fields are linearly interpolated on the basis of the nodal values.
- both displacements and stresses are continuous on the element interfaces. Nevertheless, if the ratio n.of elements / n.of vertices is 2, the model requires 2.5 variables per element, and the algebra of the problem is very simple. This allows the use of fine meshes.
- It may be convenient to combine 4 triangular elements into a single quadrangular element (the quadrangle is divided along the two diagonals). In this case we have 5 variables (2 displacements and 3 stresses) for each element.
- A convenient variant is the assumption of a constant stress within the influence area of each node.

4.4 Flux-law elements

- These are, usually, quadrangular elements (but triangular versions are also possible), endowed with topological regularity (regular geometry is even better).
- Stress and displacement variables are associated to the total flux which passes through the interfaces.
- The balance equations (equilibrium and kinematic compatibility) are written in absolute form (rigid body equilibrium and mass conservation law). The discretization error is related to the internal stress interpolation used to define the discrete form of the strain energy.
- We have 6 variables per element (2 displacements and 4 stresses). The element is suitable for fine meshes.
- The algebra is very simple for regular geometry (rectangular elements)

4.5 HC elements

- HC stands for *High Continuity* (of the displacement field).
- These elements are quadrilateral, and require a regular topology (regular geometry is even better).
- The displacement is interpolated by means of biquadratic functions, using control nodes located at the center of the element. The fulfillment of the displacement and displacement gradient continuity allows to reduce the total number of variables to only one node per element (hence there are two displacement variables).
- The (almost) compatible element type require 5 stress (Gauss) points per element. A mixed version can be defined with stress nodes located at the vertices of elements (the nodes of the mesh).
- The overall interpolation allows C^1 continuity, using a minimum number of displacement variables. So this element is suitable for very fine meshes.
- The element algebra is very easy for regular meshes (rectangular elements). However it becomes rather complex for meshes described in curvilinear coordinates.
- The element is particularly convenient for the analysis of regular domains, using regular meshes.
- It is suitable for large displacement analysis.

4.6 Flat elements

- This element is a mixed, triangular one, and is used for plates and shells.
- For the displacements a linear interpolation is used, while stresses are constant over the element. The flexural strain is taken into account introducing the relative rotations on the element edges.
- The element is quite rough, nevertheless it has a very easy algebra. It requires very fine discretizations.
- It is suitable for large displacement analysis.
- Different mixed variants are possible, using different interpolations for the stress (linear over the element, constant on the nodal area, localized on the edges).
- It is suitable for analysis based on explicit algorithms (e.g., pseudo-dynamical simulation) or when the solution is obtained using refined meshes (e.g., multigrid solutions).

4.7 Super-convergence, Multigrid solutions and much more...

- The use of appropriate elements, fine meshes, regular topology and regular (at least, nearly regular) geometry leads to a phenomenon called *super-convergence*.

That is, the errors produced in the single elements compensate each other on the overall mesh, so that the total error is smaller than the expected one (the magnitude order can be relevant, e.g., from h^2 to h^6).

- Nevertheless, the use of *really fine* meshes implies a high number of variables and this, even if a very simple algebra is involved for the single element, can make the overall analysis extremely expensive, if traditional solution strategies (based on explicit assemblage and Gauss decomposition of the stiffness matrix) are used. This last operation alone, for a problem with n variables, would involve $O(n^2)$ computational arithmetic operations.
- In situations like these, *adaptive multigrid* strategies are very useful. They are based on the simultaneous use of a sequence of discretization meshes increasingly refined, and allow to catch the solution which corresponds to the finest mesh, working essentially on the coarse ones. We obtain therefore a very powerful tool, which can solve the problem performing less than $O(n)$ operations, that is to say with an efficiency of some order greater of the one obtained through traditional procedures.
- An efficiency even greater is obtained using adaptive (selective) refinement (the solution process automatically controls the refinement).