Chapter 7

Plasticity

7.1 Introduction

Various plasticity models of mechanics are developed to describe a class of permanent deformations. These deformations are generated during loading processes and remain after the removal of the load. In this Chapter we present a few aspects of the classical linear plasticity. This model is based on the assumption on the additive separation of elastic and plastic deformation increments. In nonlinear models it is the deformation gradient $\mathbf{F}$ in which these permanent deformations are separated

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{7.1}$$

where the plastic deformation is described by $\mathbf{F}^p$ and the elastic part is $\mathbf{F}^e$. Only the product of these two objects is indeed the gradient of the function of motion $\mathbf{f}$. Neither $\mathbf{F}^e$ nor $\mathbf{F}^p$ can be written in such a form – they are not integrable. In spite of this problem, material vectors transformed by $\mathbf{F}^p$ form a vector space for each material point $\mathbf{X} \in \mathcal{B}_0$ and these spaces are sometimes called intermediate configurations. We shall not elaborate these issues of nonlinear models[1]. However, it should be mentioned that the assumption (7.1) indicates the additive separation of increments of deformation in the linear model. Namely, the time derivative of the deformation gradient has, obviously, the form

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p, \tag{7.2}$$

which yields for small strains

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}^e + \dot{\mathbf{e}}^p, \tag{7.3}$$

where $\dot{\mathbf{e}}^e$ is the elastic strain rate and $\dot{\mathbf{e}}^p$ is the plastic strain rate. In some older models it is even assumed that this additive decomposition concerns strains themselves: $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$ which is obviously much stronger than (7.3) and yields certain general doubts.

[1] e.g. see: Albrecht Bertram; *Elasticity and Plasticity of Large Deformations*, Springer Berlin, 2008.

The aim of the elastoplastic models in the displacement formulation is to find the displacement vector $\mathbf{u}$ whose gradient defines the strain $\mathbf{e}$ – as in the case of linear elasticity, and the plastic strain $\mathbf{e}^p$ which becomes an additional field.

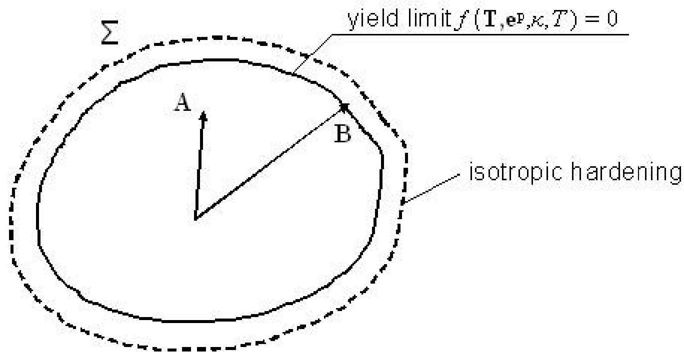

Fig. 7.1: States of material in the stress space Σ. A – the elastic state, B – the plastic state.

The most fundamental characteristic feature of classical plasticity is the distinction of an elastic domain in the space of stresses $\Sigma = \{\mathbf{T}\}, \mathbf{T} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. All paths of stresses which lie in the elastic domain produce solely elastic deformations, i.e. after inverting the process of loading the material returns to its original state. This is schematically shown in Fig. 7.1.

The elastic domain lies within the bounding yield surface also called the yield limit or the yield locus. Stress states which lie beyond this limit are attainable only by moving the whole yield surface. Such processes are called hardening. In Fig. 7.1. we demonstrate the so-called isotropic hardening. We return to this notion in the sequel. Increments of plastic strains are described by stresses whose direction points in the outward direction of the yield surface. This is related to the so-called Drucker[2] stability postulate which we present further.

The above described way of construction of plasticity is sometimes called stress space formulation and it was motivated by properties of metals. There is an alternative which has grown up from soil mechanics[3]. Such materials as rocks, soils and concrete reveal softening behaviour which violates Drucker's postulate. In order to avoid this problem, the so-called strain space formulation[4] was developed in which, instead of Drucker's postulate one applies the Ilyushyn[5] postulate. The detailed discussion of these stability problems can be found, for instance, in the book of Wu [24].

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[2] D. C. Drucker; A more fundamental approach to plastic stress-strain relations, in: *Proc. 1st Nat. Congress Appl. Mech., ASME,* 487, 1951.

[3] This formulation has been initiated by the work: Z. Mróz; Non-associated flow laws in plasticity, *Journ. de Mecanique*, **2**, 21-42, 1963.

[4] J. Casey, P. M. Naghdi; On the nonequivalence of the stress space and strain space formulations of plasticity theory, *J. Appl. Mech.*, **50**, 350, 1983.

[5] A. A. Ilyushin; On the postulate of plasticity, PMM, **25**, 503, 1961.

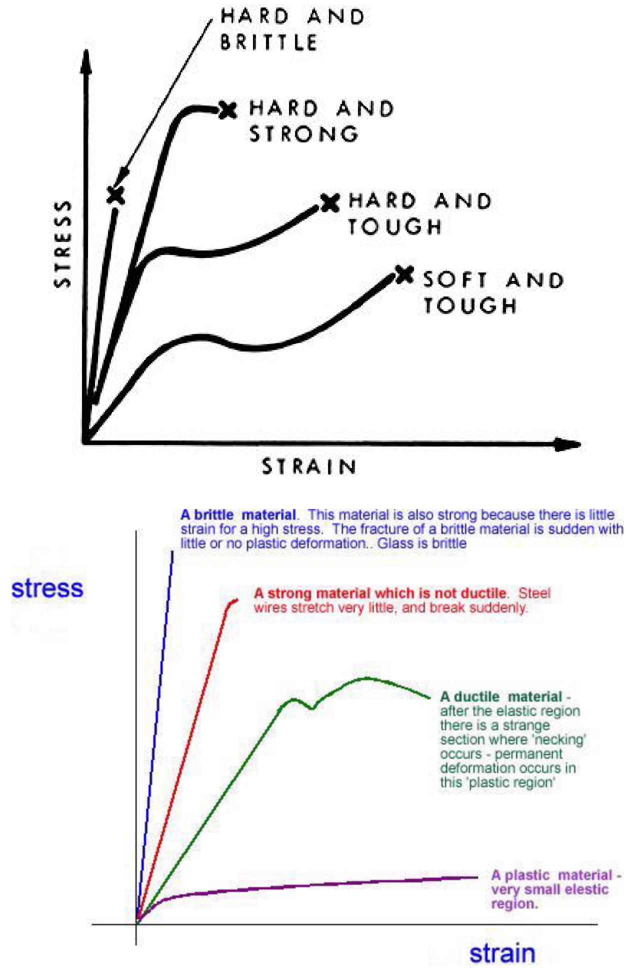

Fig. 7.2: Schematic plastic behaviour of various materials

In Fig. 7.2. we show schematically strain-stress curves for different types of materials. The steepest curve in both pictures correspond to the so-called brittle materials which practically do not reveal any plastic deformations prior to failure. Their deformations under high loading are small and they absorb only a little energy before breaking. It should be underlined that many materials may behave this way in low temperatures whereas their properties are very different in high temperatures. This transition explains mysterious catastrophes of Liberty ships in 40th of the XXth century.

The curves for ductile materials in Fig. 7.2. correspond to materials for which the classical plasticity was developed. They possess relative large irreversible deformations and by failure absorb a large amount of energy. Therefore they are called tough.

Many damages and accidents of cargo vessels were occurred, and especially for Liberty Ships. The vast majority of the sea accidents were related to brittle fracture. By 1st of April 1946, 1441 cases of damage had been reported for 970 cargo vessels, 1031 of which were to Liberty Ships. Total numbers of 4720 damages were reported. Seven ships were broken in two, e.g. "Schenectady".

7.2 Plasticity of ductile materials

We proceed to specify the yield surface in the stress space. As mentioned above this stress formulation was primarily motivated by plastic deformations of metals. In such materials the pressure p has practically no influence on plastic strains which means that the yield surface should be described only by the stress deviator: $\sigma_{ij}^D = \sigma_{ij} + p\delta_{ij},\ p = -\frac{1}{3}\sigma_{kk}$. The eigenvalues of the stress deviator follow from the eigenvalue problem

$$\left(\sigma_{ij}^D - s\delta_{ij}\right) n_j = 0, \tag{7.4}$$

and the solutions must satisfy the condition

$$I_s = s^{(1)} + s^{(2)} + s^{(3)} = 0. \tag{7.5}$$

The eigenvalues $s^{(\alpha)}$ and the eigenvalues $\sigma^{(\alpha)}$ of the full stress tensor σ_{ij} are, of course, connected by the relation

$$\sigma^{(\alpha)} = s^{(\alpha)} - p, \quad \alpha = 1, 2, 3. \tag{7.6}$$

For the purpose of formulation of various hypotheses for the yield surface, it is convenient to calculate invariants of the stress deviator and the maximum shear stresses. As presented in Subsection 3.2.4 (compare the three-dimensional Mohr circles), the extremum values of shear stress are given by the differences of three principal values of the stress tensor (radii of Mohr's circles)

$$\tau^{(1)} = \frac{\sigma^{(2)} - \sigma^{(3)}}{2}, \quad \tau^{(2)} = \frac{\sigma^{(1)} - \sigma^{(3)}}{2}, \quad \tau^{(3)} = \frac{\sigma^{(1)} - \sigma^{(2)}}{2}. \tag{7.7}$$

Hence, we have as well

$$\tau^{(1)} = \frac{s^{(2)} - s^{(3)}}{2}, \quad \tau^{(2)} = \frac{s^{(1)} - s^{(3)}}{2}, \quad \tau^{(3)} = \frac{s^{(1)} - s^{(2)}}{2}. \tag{7.8}$$

Further we use the sum of squares of these quantities. In terms of invariants of the stress tensor and of the stress deviator it has the form

$$\sum_{\alpha=1}^{3}\left(\tau^{(\alpha)}\right)^2=\frac{3}{2}\left[\frac{1}{3}I_\sigma^2-II_\sigma\right]=-\frac{3}{2}II_s=\frac{1}{2}\sigma_{eq}^2,\quad \sigma_{eq}=\sqrt{\frac{3}{2}\sigma_{ij}^D\sigma_{ij}^D},\qquad (7.9)$$

where $\sigma_{eq}=\sigma^{(1)}$ in the uniaxial tension/compression for which $\sigma^{(2)}=\sigma^{(3)}=0$. It is clear that the second invariant of the deviatoric stresses must be negative. These relations follow from the definitions of the invariants

$$\begin{aligned} I_\sigma &= \sigma_{kk}=-3p,\quad II_\sigma=\frac{1}{2}\left(I_\sigma^2-\sigma_{ij}\sigma_{ij}\right),\quad III_\sigma=\det\left(\sigma_{ij}\right), \\ I_s &= 0,\quad II_s=-\frac{1}{2}\sigma_{ij}^D\sigma_{ij}^D=-J_2=-\frac{\sigma_{eq}^2}{3},\quad III_s=J_3=\det\left(\sigma_{ij}^D\right). \end{aligned}\qquad (7.10)$$

The quantity $\sigma_{eq}=\sqrt{3J_2}$ is called the equivalent (effective) stress.

Now, we are in the position to define the elastic domain in the space of stresses. It is convenient to represent it by a domain in the three-dimensional space of principal stresses. In this space we choose the principal stresses $\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}$ as coordinates. The assumption that the pressure does not influence plastic strains means that yield surfaces in this space must be cylindrical surfaces with generatrix perpendicular to surfaces $s^{(1)}+s^{(2)}+s^{(3)}=0$, i.e. $\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}+3p=0$. The axis of those cylinders is, certainly, the straight line $\sigma^{(1)}=\sigma^{(2)}=\sigma^{(3)}$. This line is called the hydrostatic axis. In general, we can write the equation of the yield surface in the form

$$f\left(J_2,J_3,e_{ij}^p,\kappa,T\right)=0,\qquad (7.11)$$

with the parametric dependence on the plastic strain e_{ij}^p, temperature T and the hardening parameter κ. We return to these parameters later. Two examples of yield surfaces, discussed further in some details, are shown in Fig. 7.3.

It is also convenient to introduce a normal (perpendicular) vector to the yield surface in the stress space given by its gradient in this space, i.e.

$$\mathbf{N}=\frac{\frac{\partial f}{\partial \mathbf{T}}}{\left|\frac{\partial f}{\partial \mathbf{T}}\right|},\quad \text{i.e.}\quad N_{ij}=\frac{\frac{\partial f}{\partial \sigma_{ij}}}{\sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}}.\qquad (7.12)$$

Then we can introduce local coordinates in which the yield function f identifies the elastic domain of the Σ-space assuming there negative values, i.e. for all elastic processes $f<0$.

We skip here the presentation of the history of the definition of yield surfaces which goes back to Galileo Galilei. There are two fundamental forms of this surface which are still commonly used in the linear plasticity of solids. The older one was proposed by H. Tresca in 1864 and it is called Tresca-Guest surface. Its equation has the form

$$\max_\alpha\left|\tau^{(\alpha)}\right|=\sigma_0\quad\Rightarrow\quad \sigma^{(1)}-\sigma^{(3)}=2\sigma_0>0,\qquad (7.13)$$

where σ_0 is the material parameter and we have ordered the principal stresses $\sigma^{(1)}\geq\sigma^{(2)}\geq\sigma^{(3)}$. It means that the beginning of the plastic deformation appears in the point of the maximum shear stress. In the space of principal stresses it is a prism of six sides and infinite length (see: Fig. 7.3.). The parameter σ_0 may be dependent on all parameters listed in the general relation (7.11).

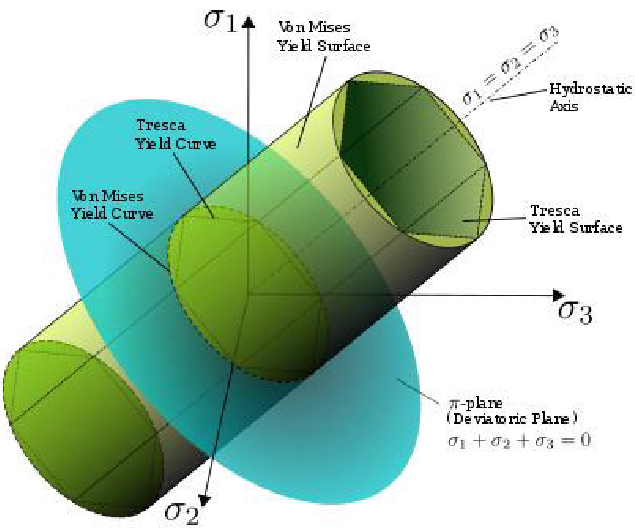

Fig. 7.3: Yield surfaces in the space of principal stresses

The second yield surface was proposed in 1904 by M. T. Huber[6] and then rediscovered in 1913 by R. von Mises and H. von Hencky. It says that the limit of elastic deformation is reached when the energy of shape changes (distortion energy) reaches the limit value $\rho\varepsilon_Y$. The distortion energy $\rho\varepsilon_D$ is defined as a part of the full energy of deformation $\rho\varepsilon$ reduced by the energy of volume changes $\rho\varepsilon_V$ (e.g. compare (5.175)). We have

$$\rho\varepsilon = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}\left(\sigma_{ij}\frac{\sigma_{kk}}{9K}\delta_{ij} + \sigma_{ij}\frac{\sigma_{ij}^D}{2\mu}\right) =$$

$$= \frac{1}{2K}\left(\frac{\sigma_{kk}}{3}\right)^2 + \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right) \quad \Rightarrow \quad \rho\varepsilon_V = \frac{p^2}{2K}, \quad \rho\varepsilon_D = \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right), \tag{7.14}$$

$$\text{i.e.} \quad \rho\varepsilon_D = \rho\varepsilon_Y \quad \Rightarrow \quad \rho\varepsilon_Y = \frac{1}{4\mu}\left(\sigma_{ij}^D\sigma_{ij}^D\right) = \frac{\sigma_{eq}^2}{6\mu}.$$

Making use of the identity, following from (7.5),

$$3\left(s^{(1)}s^{(2)} + s^{(1)}s^{(3)} + s^{(2)}s^{(3)}\right) = -\frac{1}{2}\left[\left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2\right], \tag{7.15}$$

[6] M. T. Huber; Przyczynek do podstaw wytrzymałości, *Czasop. Techn.*, Lwów, **22**, 1904. Due to the publication of this work in Polish it remained unknown until the hypothesis was rediscovered by von Mises and von Hencky.

we obtain

$$\sigma_{ij}^D \sigma_{ij}^D = \frac{1}{3}\left[\left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2 + \left(s^{(1)} - s^{(2)}\right)^2\right]. \tag{7.16}$$

Consequently, bearing (7.6) in mind, the yield limit is reached when the principal stresses fulfil the condition

$$3\sigma_{ij}^D \sigma_{ij}^D = \left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2 = 2\sigma_Y^2, \tag{7.17}$$
$$\text{i.e.} \quad \sigma_{eq} = \sigma_Y,$$

where

$$\sigma_Y = \sqrt{6\mu\rho\varepsilon_Y}, \tag{7.18}$$

is the yield limit ($\sigma^{(1)} = \sigma_Y$ in uniaxial tension, i.e. for $\sigma^{(2)} = \sigma^{(2)} = 0$). It means that for processes in which $\sigma_{eq} < \sigma_Y$ all states are elastic ($f < 0$) and otherwise the system develops plastic deformations. Clearly, the relation (7.17) defines a circular cylinder in the space of principal stresses. Its axis is again identical with the line $\sigma^{(1)} = \sigma^{(2)} = \sigma^{(3)}$, it is extended to infinity and it has common generatrix with the prism of Tresca as shown in Fig. 7.3.

★In order to compare analytically both definitions of the yield surface we show that the yield stress σ_Y calculated by means of the distortion energy of the Huber-Mises-Hencky hypothesis (7.17) is not bigger than the material parameter $2\sigma_0$ of the Tresca hypothesis (7.13). Let us write (7.17) in the following form

$$\begin{aligned}
\sigma_Y &= \frac{1}{\sqrt{2}}\sqrt{\left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2} = \\
&= \frac{\left|\sigma^{(1)} - \sigma^{(3)}\right|}{\sqrt{2}}\sqrt{\left(\frac{\sigma^{(1)} - \sigma^{(2)}}{\sigma^{(1)} - \sigma^{(3)}}\right)^2 + \left(\frac{\sigma^{(2)} - \sigma^{(3)}}{\sigma^{(1)} - \sigma^{(3)}}\right)^2 + 1} = \\
&= \frac{\left|\sigma^{(1)} - \sigma^{(3)}\right|}{\sqrt{2}}\sqrt{\left(\frac{1 - \mu_\sigma}{2}\right)^2 + \left(\frac{1 + \mu_\sigma}{2}\right)^2 + 1} = \\
&= \left|\sigma^{(1)} - \sigma^{(3)}\right|\sqrt{\frac{3 + \mu_\sigma^2}{4}},
\end{aligned} \tag{7.19}$$

where

$$\mu_\sigma = \frac{2\sigma^{(2)} - \left(\sigma^{(1)} + \sigma^{(3)}\right)}{\sigma^{(1)} - \sigma^{(3)}}, \tag{7.20}$$

is the so-called Lode parameter which describes an influence of the middle principal stress $\sigma^{(2)}$. Obviously $-1 \leq \mu_\sigma \leq 1$ which corresponds to $\sigma^{(2)} = \sigma^{(3)}$ for the lower bound, and $\sigma^{(2)} = \sigma^{(1)}$ for the upper bound. It plays an important role in the theory of civil engineering structures. Hence

$$\sigma_Y \leq \left|\sigma^{(1)} - \sigma^{(3)}\right| = 2\sigma_0.♣ \tag{7.21}$$

★ We demonstrate on a simple example an application of the notion of the yield stress. We consider a circular ring of a constant thickness with external and internal radii a and b, respectively, and an external loading by the pressures p_a and p_b on these circumferences. We check when the material of the ring reaches in all points the yield stress according to the Huber-Mises-Hencky hypothesis. This is the so-called state of the load-carrying capacity of this structure.

This is the axial symmetric problem of plane stresses. Consequently, the principal stresses in cylindrical coordinates are given by $\sigma^{(1)} = \sigma_{rr}$, $\sigma^{(2)} = \sigma_{\theta\theta}$, $\sigma^{(3)} = 0$. These components of stresses must fulfil the equilibrium condition (momentum balance (5.43))

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{7.22}$$

and, according to (7.17), at each place of the ring

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{rr})^2 + (\sigma_{\theta\theta})^2 = 2\sigma_Y^2. \tag{7.23}$$

By means of (7.22) we eliminate the component $\sigma_{\theta\theta}$ of stresses and obtain the following equation

$$\left(r\frac{ds}{dr}\right)^2 + s\left(r\frac{ds}{dr}\right) + s^2 - 1 = 0, \quad s = \frac{\sigma_{rr}}{\sigma_Y\sqrt{2}}. \tag{7.24}$$

Solution of this quadratic equation with respect to the derivative ds/dr yields

$$\frac{ds}{dr} = -\frac{s}{2r} \pm \frac{1}{2r}\sqrt{4 - 3s^2}. \tag{7.25}$$

Consequently

$$\frac{ds}{-s \pm \sqrt{4 - 3s^2}} = \frac{dr}{2r}. \tag{7.26}$$

As we have to require $|s| < 2/\sqrt{3}$ we can change the variables

$$s = \frac{2}{\sqrt{3}}\sin\varphi, \tag{7.27}$$

and this yields

$$-\frac{d\varphi}{\tan\varphi \mp \sqrt{3}} = \frac{dr}{2r}. \tag{7.28}$$

Hence, we obtain two solutions but only one of them is real and it has the form

$$r = C\sqrt{\frac{1 + \frac{3s^2}{4 - 3s^2}}{\sqrt{\frac{s\sqrt{3}}{4 - 3s^2} + \sqrt{3}}}}\exp\left[-\frac{\sqrt{3}}{2}\arctan\frac{s\sqrt{3}}{\sqrt{4 - 3s^2}}\right], \tag{7.29}$$

where C is the constant of integration.

The construction of solution is shown in Fig. 7.4. in arbitrary units. For the radius b the value of the radial stress is given by $\sigma_r = -p_b$. We adjust the curve described by the relation (7.29) in such a way that it intersects the point $(-p_b, b)$ indicated by the circle in Fig. 7.4. This yields the value of the constant C in the solution. Then for the given value of the radius a we find the value of the pressure p_a which yields the limit value for the load of this structure, i.e. its load-carrying capacity.

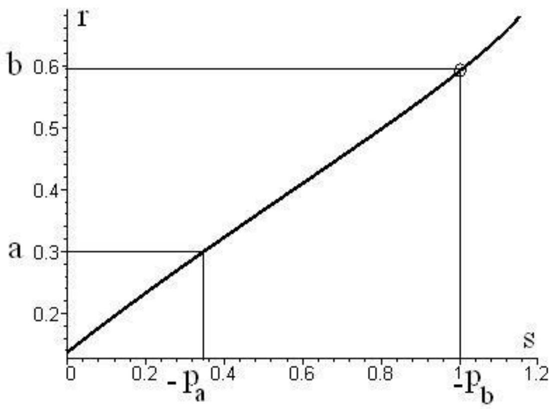

Fig. 7.4: Construction of solution for the load-carrying capacity of the ring♣

Yield surfaces impose conditions on elastic solutions under which the system possesses only elastic strains. In some design problems this is already sufficient. However, many engineering structures admit some plastic deformations – for example, in the case of concrete it is the rule – and then we have to find a way to describe the evolution of plastic strains $\mathbf{e}^p$. We proceed to develop such models.

First of all, we have to define not only the shape of the yield surface in the stress space, as we did above, but also its dependence on parameters listed in (7.11). We indicate here a few important examples. It is convenient to write the yield function in the form

$$f\left(J_2, J_3, e^p_{ij}, \kappa, T\right) = F\left(J_2, J_3\right) - \sigma_Y\left(e^p_{eq}, \kappa, T\right) = 0, \tag{7.30}$$

where σ_Y is the yield limit in the uniaxial tension/compression test for which

$$\begin{aligned} \sigma_{22} &= \sigma_{33} = \sigma^{(2)} = \sigma^{(3)} = 0 \quad \Rightarrow \quad \sigma_{eq} = \sigma^{(1)} = \sigma_{11}, \\ \dot{e}^p_{11} &= -2\dot{e}^p_{22} = -2\dot{e}^p_{33}, \quad \dot{e}^p_{ij} = 0 \text{ for } i \neq j \quad \Rightarrow \quad \dot{e}^p_{eq} = \dot{e}^p_{11}. \end{aligned} \tag{7.31}$$

where σ_{eq} is given by (7.9), and it is assumed to be given in terms of arguments listed in (7.30). e^p_{eq} is the equivalent plastic strain obtained from the integration in time of the effective rate of plastic deformation

$$\dot{e}^p_{eq} = \sqrt{\frac{2}{3}\dot{e}^p_{ij}\dot{e}^p_{ij}}. \tag{7.32}$$

Obviously, for isotropic materials we expect $\dot{e}_{ij}^{p}$ to be deviatoric. This yields relations (7.31).

Let us begin with the simplest case. In a particular case of ideal plasticity we consider materials without hardening. Then the function (7.30) has the form

$$f = \sqrt{\frac{3}{2}\sigma_{ij}^{D}\sigma_{ij}^{D}} - \sigma_Y = 0, \tag{7.33}$$

with the constant yield limit σ_Y. Of course, the stress tensor must be such that elastic processes remain within the elastic domain which is characterized by $f < 0$. Plastic deformations may develop for stresses which belong to the yield surface. Their changes yielding plastic deformation must remain on this surface which means that the increments of stresses described by $\dot{\sigma}_{ij}$ must be tangent to the yield surface. Hence, for such processes

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = 0, \tag{7.34}$$

as the gradient $\partial f/\partial\sigma_{ij}$ is perpendicular to the yield surface.

Now we make the fundamental constitutive assumption, specifying the rate of plastic strain $\dot{e}_{ij}^{p}$ and require that this quantity follows from a potential $G(\sigma_{ij})$ defined on the stress space

$$\dot{e}_{ij}^{p} = \dot{\lambda}\frac{\partial G}{\partial \sigma_{ij}}, \tag{7.35}$$

where $\dot{\lambda}$ is a scalar function following from the so-called Prager consistency condition. We demonstrate it further.

Let us introduce the notion of the outward normal vector to the surface $f = 0$ (compare (7.12)). Clearly

$$N_{ij} = \frac{\partial f}{\partial \sigma_{ij}}\left\|\frac{\partial f}{\partial \sigma_{kl}}\right\|^{-1}, \quad \left\|\frac{\partial f}{\partial \sigma_{kl}}\right\| = \sqrt{\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}}, \tag{7.36}$$

is such a vector. For the yield surface (7.33) it becomes

$$N_{ij} = \frac{\sigma_{ij}^{D}}{\left\|\sigma_{kl}^{D}\right\|}, \quad \left\|\sigma_{kl}^{D}\right\| = \sqrt{\sigma_{kl}^{D}\sigma_{kl}^{D}} = \sqrt{\frac{2}{3}}\sigma_Y. \tag{7.37}$$

Obviously, we have the following classification

$$\dot{e}_{ij}^{p} = \begin{cases} 0 \quad \text{for} \quad f < 0 \text{ or } f = 0 \text{ and } N_{ij}\dot{\sigma}_{ij} < 0. \\ \dot{\lambda}\frac{\partial G}{\partial \sigma_{ij}} \quad \text{for} \quad f = 0 \text{ and } N_{ij}\dot{\sigma}_{ij} = 0. \end{cases} \tag{7.38}$$

The condition $N_{ij}\dot{\sigma}_{ij} < 0$ means that the process yields the unloading – as N_{ij} is orthogonal to the yield surface, $\dot{\sigma}_{ij}$ must point in the direction of the elastic domain and, consequently, the process must be elastic.

In a particular case when the potential G and the yield function are identical we obtain

$$\dot{e}^p_{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}. \tag{7.39}$$

This is the so-called associated flow rule. In this case, the rate of plastic deformation $\dot{e}^p_{ij}$ is perpendicular to the yield surface, i.e. it is parallel to the normal vector N_{ij}.

It is appropriate to make here the following remark concerning the geometry of the yield surface. If this surface were not convex then in points in which it is concave tangent changes of the stress tensor $\dot{\sigma}_{ij}$, i.e. $(\partial f / \partial \sigma_{ij})\, \dot{\sigma}_{ij} = 0$, would yield stresses in the interior of the yield surface, i.e. in the range $f < 0$. This would be related to the development of pure elastic deformations in contrast to the assumption that tangent changes of stress yield plastic deformations. Therefore, in the classical plasticity nonconvex yield surfaces are not admissible. This is the subject of the so-called Drucker stability postulate. In the local form for the associated flow rules (7.39) it can be written as

$$\dot{e}^p_{ij} \dot{\sigma}_{ij} > 0. \tag{7.40}$$

It is also sometimes postulated in the global form

$$\int \left(\sigma_{ij} - \sigma^0_{ij} \right) de^p_{ij} > 0, \tag{7.41}$$

which shows that the postulate imposes a restriction on the work of plastic deformations between an arbitrary initial state of stress σ^0_{ij} and an arbitrary finite state of stress σ_{ij}.

For the Huber-Mises-Hencky yield function (7.33) we obtain the associated flow rule

$$\dot{e}^p_{ij} = \dot{\lambda} \sqrt{\frac{3}{2}} \frac{\sigma^D_{ij}}{\left\| \sigma^D_{kl} \right\|} = \dot{\lambda} \sqrt{\frac{3}{2}} N_{ij}. \tag{7.42}$$

In the simple uniaxial tension/compression test we have then

$$\dot{e}^p_{eq} = \dot{e}^p_{11} = \dot{\lambda} \quad \Rightarrow \quad \dot{e}^p_{ij} \sigma_{ij} = \dot{\lambda} \sqrt{\frac{3}{2}} \left\| \sigma^D_{kl} \right\| = \dot{\lambda} \sigma_{11} = \dot{e}^p_{eq} \sigma_Y. \tag{7.43}$$

Hence for the rate of work (working) we obtain

$$\begin{aligned} \dot{W} = \dot{e}_{ij} \sigma_{ij} = \left(\dot{e}^e_{ij} + \dot{e}^p_{ij} \right) \sigma_{ij} \quad &\Rightarrow \\ \Rightarrow \quad \dot{W}_p = \dot{e}^p_{ij} \sigma_{ij} = \dot{\lambda} \sqrt{\frac{3}{2}} \left\| \sigma^D_{kl} \right\| = \dot{e}^p_{eq} \sigma_{eq} &= \dot{e}^p_{eq} \sigma_Y. \end{aligned} \tag{7.44}$$

The last expression – $\dot{e}^p_{eq} \sigma_Y$ – describes the plastic working in the one-dimensional test which is an amount of energy dissipated by the system per unit time due to the plastic deformation. Hence

$$\dot{\lambda} \geq 0, \tag{7.45}$$

and the equality holds only for elastic deformations. This statement follows easily from the second law of thermodynamics.

In order to construct an equation for plastic strains we account for the additive decomposition (7.3). For $f < 0$ we have elastic processes and then this property indicates (compare (5.26)) the following Prandl-Reuss equation for the rate of deformation

$$\dot{e}_{ij} = \left(\frac{\dot{\sigma}_{ij}^D}{2\mu} + \frac{\dot{\sigma}_{kk}}{9K}\delta_{ij} \right) + \dot{\lambda}\sqrt{\frac{3}{2}} N_{ij} = \tag{7.46}$$

$$= \left(\frac{\dot{\sigma}_{ij}^D}{2\mu} + \frac{\dot{\sigma}_{kk}}{9K}\delta_{ij} \right) + \dot{W}_p \frac{\sigma_{ij}^D}{\left\| \sigma_{kl}^D \right\|^2}. \tag{7.47}$$

This follows from the property of isotropic elastic materials for which the eigenvectors for the stress and for the strain are identical[7]. The spherical part is, obviously, purely elastic while the deviatoric part can be written in the form

$$\frac{d\sigma_{ij}^D}{dt} + \dot{W}_p \frac{4\mu}{3\sigma_Y^2} \sigma_{ij}^D = 2\mu \frac{de_{ij}^D}{dt}. \tag{7.48}$$

This equation is very similar to the evolution equation for stresses within the standard linear model of viscoelasticity (6.58) divided by the relaxation time τ. However, there is a very essential difference between these two models. It is easy to check that the equation of viscoelasticity (6.58) is not invariant with respect to a change of time scale $t \to \alpha t$, where α is an arbitrary constant. This indicates the rate dependence in the reaction of the material. It is not the case for the equation of plasticity (7.48). Differentiation with respect to time appears in all terms of this equation and for this reason the transformation parameter α cancels out. This is the reason for denoting the consistency parameter by $\dot{\lambda} = de_{eq}^p/dt$. It transforms: $t \to \alpha t \Rightarrow \dot{\lambda} \to \dot{\lambda}/\alpha$. Therefore, the classical plasticity is rate independent. The response of the system is the same for very fast and very slow time changes of the loading. In reality, metals do possess this property when the rate of deformation $\dot{e}_{eq}^p$ is approximately $10^{-6} - 10^{-4}$ 1/s. For higher rates one has to incorporate the rate dependence (compare the book of Lemaitre, Chaboche [9] for further details). This yields viscoplastic models presented further in these notes.

In the more general case of isotropic hardening and for isothermal processes σ_Y becomes a function of e_{eq}^p alone. For many materials it is also important to include the temperature dependence. Then σ_Y becomes the function of these two quantities. The model is similar to this which we have considered above but one has to correct the definition of the consistency parameter $\dot{\lambda}$. Finally, a dependence on the hardening parameter κ means that we account for the accumulation of plastic deformations in the material. The most common definitions of this parameter are as follows

a) the parameter accounting for the accumulation of the plastic energy

$$\kappa = \int_0^t \sigma_{ij}(\xi) \, \dot{e}_{ij}^p(\xi) \, d\xi, \tag{7.49}$$

[7] In order to prove it, it is sufficient to compare the eigenvalue problems for deviatoric stress and strain tensors.

b) Odqvist parameter accounting for the accumulation of the plastic deformation (compare (7.30) and (7.32))

$$\kappa = \int_0^t \sqrt{\frac{2}{3}\dot{e}^p_{ij}\dot{e}^p_{ij}}d\xi = \int_0^t \frac{de^p_{eq}}{d\xi}d\xi. \tag{7.50}$$

Then the relation (7.11) yields the consistency condition

$$\dot{f} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial e^p_{ij}}\dot{e}^p_{ij} + \frac{\partial f}{\partial T}\dot{T} + \frac{\partial f}{\partial \kappa}\dot{\kappa} = 0. \tag{7.51}$$

Simultaneously, $\dot{e}^p_{ij} \neq 0$ only in processes of loading which are defined by the relation

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} > 0. \tag{7.52}$$

In the limit case

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} = 0, \tag{7.53}$$

we say that the process is neutral. Finally, for the process of unloading,

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} < 0. \tag{7.54}$$

Consequently, for the evolution of plastic deformation we have the following relations

$$\dot{e}^p_{ij} = \begin{cases} 0 \quad \text{for either } f < 0 \text{ or } f = 0 \text{ and } \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} \leq 0 \\ \dot{\lambda}\dfrac{\partial f}{\partial \sigma_{ij}} \quad \text{for } f = 0 \text{ and } \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial T}\dot{T} > 0. \end{cases} \tag{7.55}$$

In the case of the hardening parameter (7.49) the consistency condition (7.51) can be written in the form

$$\dot{f} = \frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T} + \left[\frac{\partial f}{\partial e^p_{ij}} + + \frac{\partial f}{\partial \kappa}\sigma_{ij}\right]\dot{e}^p_{ij} = 0.$$

Hence, we obtain the following relation for the consistency parameter

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T}}{D}, \quad D = -\frac{\partial f}{\partial e^p_{ij}}\frac{\partial f}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \kappa}\frac{\partial f}{\partial \sigma_{ij}}\sigma_{ij}. \tag{7.56}$$

The quantity D is called the hardening function. We have

$$\dot{\lambda} > 0 \quad \Rightarrow \quad D > 0. \tag{7.57}$$

The flow rule can be now written in the form

$$\dot{e}^p_{ij} = \frac{1}{D}\frac{\partial f}{\partial \sigma_{ij}}\left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial T}\dot{T}\right). \tag{7.58}$$

The right hand side is the homogeneous function of $\dot{\sigma}_{ij}$ and $\dot{T}$. Consequently, this flow rule possesses the same time invariance as the rule (7.39) for the model without hardening, i.e. the model is rate independent.

Apart from the above presented isotropic hardening materials reveal changes of the yield limit which can be attributed to the shift of the origin in the space of stresses. A typical example is the growth of the yield stress in tensile loading with the simultaneous decay of the yield stress for compression. In the uniaxial case it means that the whole stress-strain diagram will be shifted on a certain value of stresses. This is the Bauschinger effect.

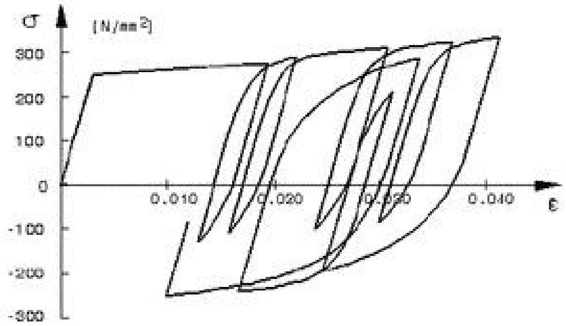

Fig. 7.5: An example of Bauschinger effect in cyclic loading

The corresponding hardening is called kinematical or anisotropic. It is described by the so-called back-stress $\mathbf{Z} = Z_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ which specifies the shift of the origin in the stress space. The yield function can be then written in the form

$$f(\mathbf{T}, \mathbf{Z}, \kappa) = F(\mathbf{T}, \mathbf{Z}) - \sigma_Y\left(e^p_{eq}, T\right) = \sqrt{\frac{3}{2}\bar{\sigma}^D_{ij}\bar{\sigma}^D_{ij}} - \sigma_Y = 0, \tag{7.59}$$

where

$$\bar{\sigma}^D_{ij} = \sigma^D_{ij} - Z_{ij}. \tag{7.60}$$

One has to specify an equation for the back-stress. It is usually assumed to have the form of the evolution equation, e.g.

$$\dot{Z}_{ij} = \dot{\beta}\left(\sigma_{ij} - Z_{ij}\right), \tag{7.61}$$

where $\dot{\beta}$ is a material parameter. We skip here the further details referring to numerous monographs on the subject[8].

[8]e.g. see the book [9] or
Albrecht Bertram; *Elasticity and Plasticity of Large Deformations*, Springer Berlin, 2008.
Michal Kleiber; *Handbook of Computational Solid Mechanics*, Springer, Heidelberg, 1998.
Gerard A. Maugin; *The Thermomechanics of Plasticity and Fracture,* Cambridge University Press, 1992.

7.3 Plasticity of soils

Theories of irrecoverable, permanent deformations of soils is very different from the plasticity of metals presented above. Metals produce plastic deformations primarily due to the redistribution and production of crystallographic defects called dislocations. Plastic behaviour of soils is mainly connected with the redistribution of grains and it is strongly influenced by fluids filling the voids (pores) of such a granular material. Strain due to the deformation of grains is often negligible in comparison to the amount of shear and dilatation caused by relative motions of grains. The behaviour is entirely different in the case of dry granular materials (frictional materials) than a material saturated by, for instance, water or oil where the cohesive forces play an important role. A detailed modern presentation of the problem of permanent deformations of soils can be found in the book of D. Muir Wood [23] (compare also a set of lectures von Verruijt [19]). Similar issues for rocks are presented in the classical book of Jaeger, Cook and Zimmerman[6]. We limit the attention only to few issues of this subject.

Attempts to describe the plasticity of granular materials stem from Coulomb, who formulated a simple relation between the normal stress σ_n on the surface with a normal vector $\mathbf{n}$ and the shear stress τ_n on this surface. It is a generalization of the law of friction between two bodies and has the form

$$|\tau_n| = c - \sigma_n \tan\varphi, \tag{7.62}$$

where φ is the so-called friction angle (angle of repose) and c denotes the cohesion intercept. This relation is called Mohr-Coulomb law. For dry granular materials the cohesion does not appear, $c = 0$, and then the angle of repose φ is the only material parameter. It is, for instance, the slope of natural sand hills and pits (Fig. 7.6).

Fig. 7.6: Sand pit trap of antlion in dry sand. Slope almost equal to φ

The above relation leads immediately to the yield function in terms of principal stresses $\sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)}$.. Namely

$$f\left(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\right) = \left(\sigma^{(1)} - \sigma^{(3)}\right) + \left(\sigma^{(1)} + \sigma^{(3)}\right)\sin\varphi - 2c\cos\varphi = 0. \tag{7.63}$$

The derivation from properties of Mohr's circle is shown in Fig. 7.7. Namely

$$|\tau_n| = \frac{\sigma^{(1)} - \sigma^{(3)}}{2} \cos\varphi, \quad \sigma_n = \frac{\sigma^{(1)} + \sigma^{(3)}}{2} - \frac{\sigma^{(1)} - \sigma^{(3)}}{2} \sin\varphi. \qquad (7.64)$$

Substitution in (7.62) yields (7.63).

For $\varphi = 0$ and $c = \sigma_0$ the yield function (7.63) becomes the Tresca-Guest yield condition (7.13).

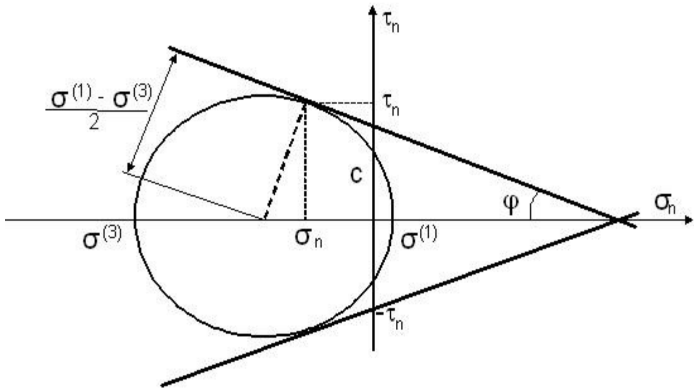

Fig. 7.7: Construction of Mohr-Coulomb yield function

Cohesive forces are influencing not only the relation between normal and shear stresses. Due to the porosity of granular materials a fluid in pores yields cohesive interactions as well as it carries a part of external loading. This observation was a main contribution of von Terzaghi to the theory of consolidation of soils[9]. He has made an assumption that the pore pressure p does not have an influence on the plastic deformation of soils. The meaning of p is here the same as in Subsection 6.4. and it should not be confused with the trace of the bulk stress σ_{ij}, i.e. $p \neq -\frac{1}{3}\sigma_{kk}$. It means that the stress appearing in yield functions must be reduced by subtracting the contribution of this pressure. If we define the effective stress

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}, \qquad (7.65)$$

then the Mohr-Coulomb yield function becomes

$$\left(\sigma'^{(1)} - \sigma'^{(3)}\right) + \left(\sigma'^{(1)} + \sigma'^{(3)}\right) \sin\varphi - 2c\cos\varphi = 0. \qquad (7.66)$$

$$\sigma'^{(\alpha)} = \sigma^{(\alpha)} + p, \quad \alpha = 1, 2, 3.$$

This function is shown in the upper panel of Fig. 7.8.

Further we distinguish by primes all quantities based on the effective stress.

[9] K. von Terzaghi; *Erdbaumechanik auf bodenphysikalischer Grundlage,* Franz Deuticke, Wien, 1925.

Incidentally, a similar notion of effective stresses appears in the theory of damage – it is related to changes of reference surface due to the appearance of cracks. Such models shall be not presented in these notes.

★**Remark.** There exists some confusion within the soil mechanics concerning the definition of positive stresses. Soils carry almost without exception only compressive loads (compare Fig. 7.7. and 7.8.) and, for this reason, in contrast to the classical continuum mechanics, a compressive stress is assumed to be positive. This is convenient in a fixed system of coordinates related to experimental setups such as triaxial apparatus. Then pressure p in the definition of effective stresses (7.65) would appear with the minus sign. Usually it is denoted in soil mechanics by u. In some textbooks[10] both conventions concerning the sign of stresses appear simultaneously. However, such a change of sign in a general stress tensor yields the lack of proper invariance with respect to changes of reference systems. It is also contradictory with the choice of the positive direction of vectors normal to material surfaces on which many mathematical problems of balance laws and the Cauchy Theorem rely. For these reasons, we work here with the same convention as in the rest of this book – tensile stress is positive.

In addition, one should be careful in the case of relation of such one-component models to models following from the theory of immiscible mixtures, for instance to Biot's model. Such models are based on partial quantities and then the pore pressure p is not the partial pressure p^F of a multicomponent model but rather $p = p^F/n$, where n is the porosity.♣

In soil mechanics, where the definitions of elastic domains described in the previous Subsection are not appropriate, it is convenient to introduce special systems of reference in the space of effective principal stresses. One of them is directly related to the set of invariants (7.10)

$$p' = \frac{1}{3}I_1' = -\frac{1}{3}I_\sigma', \quad q = \sqrt{3J_2'} = \sigma_{eq}, \quad r = 3\sqrt[3]{\frac{J_3'}{2}}. \tag{7.67}$$

Another one is a cylindrical system. One of the axes is the pressure $\left(-\frac{1}{3}\sigma'_{kk}\right)$, i.e. it is oriented along the line $\sigma'^{(1)} = \sigma'^{(2)} = \sigma'^{(3)}$. It is denoted by ξ and scaled $\xi = I_\sigma'/\sqrt{3}$. The other two coordinates are defined by the relations

$$\rho = \sqrt{2J_2'} \equiv \sqrt{\frac{2}{3}}\sigma'_{eq} \equiv \sqrt{\sigma'^D_{ij}\sigma'^D_{ij}}, \quad \cos(3\theta) = \left(\frac{r}{\sigma'_{eq}}\right)^3. \tag{7.68}$$

These are the so-called Haigh–Westergaard coordinates. The (ξ, ρ)- planes are called Rendulič planes and the angle θ is called the Lode angle. The transformation from these coordinates back to principal stresses is given by the relation

$$\begin{pmatrix} \sigma'^{(1)} \\ \sigma'^{(2)} \\ \sigma'^{(3)} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} \xi \\ \xi \\ \xi \end{pmatrix} + \sqrt{\frac{2}{3}}\rho \begin{pmatrix} \cos\theta \\ \cos\left(\theta - \frac{2}{3}\pi\right) \\ \cos\left(\theta + \frac{2}{3}\pi\right) \end{pmatrix}. \tag{7.69}$$

Mohr-Coulomb yield function in the Haigh–Westergaard coordinates has the following form

$$\left[\sqrt{3}\sin\left(\theta + \frac{\pi}{3}\right) - \sin\varphi\cos\left(\theta + \frac{\pi}{3}\right)\right]\rho - \sqrt{2}\xi\sin\varphi = \sqrt{6}c\cos\varphi. \tag{7.70}$$

[10] e.g. [23] or R. Lancellotta; *Geotechnical Engineering,* Balkema, Rotterdam, 1995.

Alternatively, in terms of the invariants (p', q, r) we can write

$$\left[\frac{1}{\sqrt{3}\cos\varphi}\sin\left(\theta+\frac{\pi}{3}\right)-\frac{1}{3}\tan\varphi\cos\left(\theta+\frac{\pi}{3}\right)\right]q-p'\tan\varphi=c, \tag{7.71}$$
$$\theta=\frac{1}{3}\arccos\left(\frac{r}{q}\right)^3.$$

As in the classical theory of plasticity, modifications of Mohr-Coulomb condition were introduced in order to eliminate corners in the yield surface. One of such modifications was introduced by D. C. Drucker and W. Prager. This condition for the limit state of soils has the following form

$$\sqrt{J_2'}-\frac{\sqrt{3}\cos\varphi}{\sqrt{3+\sin^2\varphi}}c-\frac{\sin\varphi}{\sqrt{3\left(3+\sin^2\varphi\right)}}I_\sigma'=0, \tag{7.72}$$

where the invariants J_2' and I_σ' are defined by relations for effective stress analogous to (7.10). This function is shown in the lower panel of Fig. 7.8. The dependence on the invariant I_σ' follows from the dependence of the yield in soils on volume changes, i.e. it describes an influence of dilatancy on the appearance of the critical limit state. For $\varphi=0$ and $c=\sigma_Y/\sqrt{3}$ this condition becomes identical with Huber-Mises-Hencky condition (7.17).

Due to its simplicity the Mohr-Coulomb yield surface is often used to model the plastic flow of geomaterials (and other cohesive-frictional materials). However, many such materials show dilatational behavior under triaxial states of stress which the Mohr-Coulomb model does not include. Also, since the yield surface has corners, it may be inconvenient to use the original Mohr-Coulomb model to determine the direction of plastic flow. Therefore it is common to use a non-associated plastic flow potential that is smooth. For example, one is using the function

$$g=\sqrt{\left(\alpha c_Y\tan\psi\right)^2+G^2\left(\varphi,\theta\right)q^2}-p'\tan\varphi, \tag{7.73}$$

where α is a parameter, c_Y is the value of c when the plastic strain is zero (also called the initial cohesion yield stress), ψ is the angle made by the yield surface in the Rendulič plane at high values of p' (this angle is also called the dilation angle), and $G\left(\varphi,\theta\right)$ is an appropriate function that is also smooth in the deviatoric stress plane.

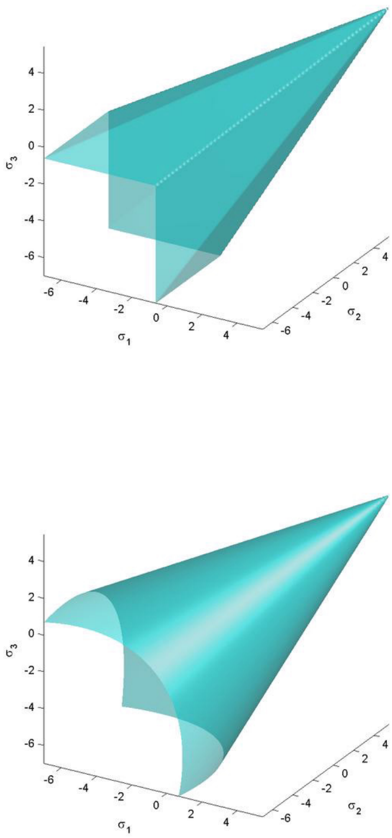

Fig. 7.8: Yield surfaces (7.66) **and** (7.72) **in the space of principal effective stresses** $\sigma_1 = \sigma'^{(1)}$, $\sigma_2 = \sigma'^{(2)}$, $\sigma_3 = \sigma'^{(3)}$.

We shall not expand this subject anymore. Due to the vast field of applications: soils, powders, avalanches, debris flows and many others, the number of models describing the critical behaviour of such materials is also very large. Cap plasticity models, Cam-Clay (CC) models, Modified-Cam-Clay (MCC) models, Mroz models, etc. are based on similar ideas as the models presented above. There exists also a class of hypoplasticity models in which the notion of the yield surface does not appear at all and which seem to fit well phenomena appearing in sands[11].

[11] compare articles of E. Bauer: Analysis of Shear Banding with a Hypoplastic Constitutive Model for a Dry and Cohesionless Granular Material, 335-350, and D. Kolymbas: The Importance of Sand in Earth Sciences, both in: B. Albers (ed.); *Continuous Media with Microstructure*, Springer, Berlin, 2010.

7.4 Viscoplasticity

There are many ways of extension of the classical plasticity to include rate dependence. Obviously, one of them would be to incorporate additionally some viscous properties as we did in Chapter 7. This kind of the model is developed since early works of P. Perzyna[12]. The other way, less ambitous, is to incorporate a rate dependence in the definition of the yield function. In principle, the classical yield function cannot exist in such models but one gets results by direct extension of plasticity models presented in this Chapter. For such models it is advocated in the books of Lemaitre, Chaboche [9] and Lemaitre, Desmorat [10].

We present here only a few hints to the model of the second kind. Namely, it is assumed that the yield criterion satisfies the relation

$$\begin{aligned} f &= 0, \quad \dot{f} = 0 \quad \text{– plasticity,} \\ f &= \sigma_V > 0 \quad \text{– viscoplasticity,} \end{aligned} \tag{7.74}$$

with $f < 0$ satisfied in the elastic domain. σ_V is a viscous stress given by a viscosity law. In both cases f can be chosen according to the rules discussed in previous Subsections. For instance, in the case of Huber-Mises-Hencky model with isotropic and kinematic hardening we have

$$\begin{aligned} f &= (\mathbf{T} - \mathbf{Z})_{eq} - \kappa - \sigma_Y, \qquad (7.75) \\ (\mathbf{T} - \mathbf{Z})_{eq} &= \sqrt{\frac{3}{2}\left(\sigma_{ij}^D - Z_{ij}^D\right)\left(\sigma_{ij}^D - Z_{ij}^D\right)}, \end{aligned}$$

where κ describes the isotropic hardening related to the size growth of the yield surface. It may be, for instance, assumed to have the exponential form

$$\kappa = \kappa_\infty \left[1 - \exp\left(-b e_{eq}^p\right)\right], \tag{7.76}$$

where κ_∞, b are material parameters depending on temperature. Sometimes a power law $\kappa = K_p \left(e_{eq}^p\right)^{1/M}$ is sufficient.

Kinematic hardening described by the back-stresses Z_{ij} requires an evolution equation. It may have the form (7.61) or it may be the so-called Armstrong-Frederick law[13]

$$\frac{d}{dt}\left(\frac{Z_{ij}}{C}\right) = \frac{2}{3}\dot{e}_{ij}^p - \frac{\gamma}{C} Z_{ij} \dot{e}_{eq}^p, \tag{7.77}$$

for which the identification of parameters is easier [10].

The viscous stress σ_V is also given by various empirical relations. Two of them have the form

1) Norton power law

$$\sigma_V = K_N \left(\dot{e}_{eq}^p\right)^{1/N}, \tag{7.78}$$

[12] e.g. P. Perzyna; The constitutive equations for the rate sensitive plastic materials, *Quart. Appl. Math.,* **20**, 321-332, 1963.

[13] P. J. Armstrong, C. O. Frederick; A mathematical representation of the multiaxial Bauschinger effect, CEGB Report, RD/B/N731, Berkeley Nuclear Laboratories, 1966.

2) exponential law leading to the saturation at large plastic rates

$$\sigma_V = K_\infty \left[1 - \exp\left(-\frac{\dot{e}^p_{eq}}{n}\right)\right], \tag{7.79}$$

where K_N, K_∞, N and n are material parameters.

In Fig. 7.9 we show a comparison of results for various viscous models[14].

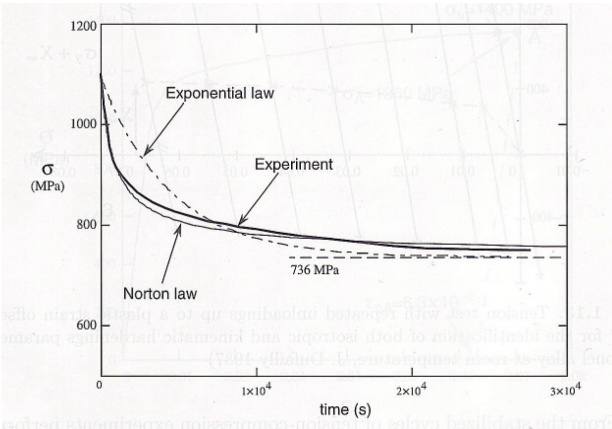

Fig. 7.9: Relaxation test for the identification of viscosity parameters – Inconel alloy at $\theta = 627^0$ C.

For the alloy investigated by Lemaitre and Dufailly the following parameters are appropriate

$$E = 160 \text{ GPa}, \quad K_N = 75 \text{ GPa/s}^{1/N}, \quad N = 2.4, \quad K_\infty = 10^4 \text{ GPa}, \quad n = 1.4 \times 10^{-2}\text{s}^{-1}.$$

Rate-dependent viscoplastic models must be used in cases of high deformation rates. For metals, the rates up to app. 10^{-3} 1/s do not influence substantially results in the plastic range of deformations. For higher rates the yield limit may grow even three times by the rate 100 1/s[15].

[14] J. Lemaitre, J. Dufailly; Damage measurements, *Engn. Fracture Mech.,* **28**, 1987

[15] P. Perzyna; *Thermodynamics of Inelastic Materials* (in Polish), PWN, Warsaw, 1978.