CHAPTER 3

FOUNDATIONS OF THE THEORY OF PLASTICITY

3.1 INTRODUCTION

This chapter describes some of the fundamental elements of the theory of plasticity. These elements include yield conditions, plastic potential and flow rule, principle of maximum plastic work, isotropic hardening and Drucker's stability postulate, kinematic and mixed hardening, and general stress-strain relations. These concepts form the foundations of the theory of plasticity. A good review of these fundamental concepts can be found in Hill (1950), Prager (1955) and Naghdi (1960). It should be stressed that the general stress-strain relations described here are only valid for small deformation, but their extension to large deformation will be covered in Chapter 14 when used in finite element analysis of large deformation problems.

3.2 YIELD CRITERION

A condition that defines the limit of elasticity and the beginning of plastic deformation under any possible combination of stresses is known as the yield condition or yield criterion. In the elastic region, all the deformation will be recovered once the applied stress is removed (i.e. unloading of stress to zero). However once the yield condition is reached, some of the deformation will be permanent in the sense that it cannot be recovered even after the stress is removed completely. This part of the deformation is known as plastic deformation and the remaining deformation is recoverable upon removal of the stress and is known as elastic deformation.

For the simple case of one-dimensional loading, the yield criterion is defined by a stress value beyond which plastic deformation will occur. In other words, the criterion of yield is graphically represented by *a point*. For the case of two dimensional loading, the yielding will occur when the combination of stresses applied in the two loading directions touches *a curve*. In the same way, for the case of three dimensional loading, plastic deformation will occur once the combination of the stresses applied in the three directions touches *a surface* (often known as a yield surface). In short, the yield criterion is generally represented by a surface in stress space. When the stress state is within the yield surface, material behavior is said to be elastic. Once the stress state is on the yield surface, plastic deformation will be produced.

Mathematically, a general form of yield criterion (or surface) can be expressed in terms of either the stress tensor or the three stress invariants as follows:

$$f(\sigma_{ij}) = f(I_1, I_2, I_3) = 0 \tag{3.1}$$

3.3 PLASTIC POTENTIAL AND PLASTIC FLOW RULE

A key question that the theory of plasticity sets out to answer is how to determine the plastic deformation (or plastic strains) once the stress state is on the yield surface. The most widely used theory is to assume that the plastic strain rate (or increment) can be determined by the following formula (von Mises, 1928; Melan, 1938; Hill, 1950):

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{3.2}$$

where $d\lambda$ is a positive scalar, and

$$g = g(\sigma_{ij}) = g(I_1, I_2, I_3) = 0 \tag{3.3}$$

is known as the plastic potential, which may or may not be the same as the yield surface. Equation (3.2) is referred to as a plastic flow rule that basically defines the ratios of the components of the plastic strain rate. This plastic flow rule was based on the observation by de Saint-Venant (1870) that for metals the principal axes of the plastic strain rate coincide with those of the stress. This is the so-called coaxial assumption, which has been the foundation of almost all the plasticity models used in engineering. It must be noted that recent experimental data suggests that the coaxial assumption is generally not valid for soils (see Chapter 8 for details).

If the plastic potential is the same as the yield surface, then the plastic flow rule (3.2) is called the associated flow (or normality) rule. Otherwise it is called non-associated flow rule. The associated flow rule follows from considerations of the plastic deformation of polycrystalline aggregates in which individual crystals deform by slipping over preferred planes (Bishop and Hill, 1951).

If the unit normal to the plastic potential approaches a finite number of linearly independent limiting values as the stress point approaches the singular point in question, Koiter (1953) proposes the following generalized flow rule

$$d\varepsilon_{ij}^p = \sum_{i=1}^{n} d\lambda_i \frac{\partial g_i}{\partial \sigma_{ij}} \tag{3.4}$$

where $d\lambda_i$ are nonnegative and $\partial g_i / \partial \sigma_{ij}$ are the linearly independent gradients.

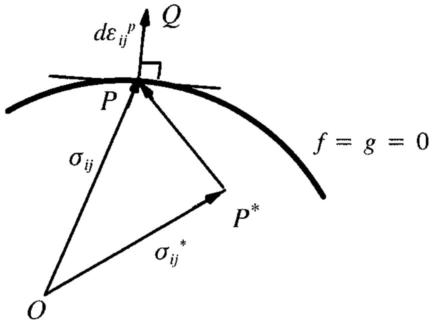


Figure 3.1: Maximum plastic work principle

3.4 PRINCIPLE OF MAXIMUM PLASTIC WORK

Suppose the plastic strain rate $d\varepsilon_{ij}^{p}$ is given and the corresponding stress state, σ_{ij}, determined from the normality rule and the yield criterion, is represented by a point P in the stress space, Figure 3.1. If σ_{ij}^{*} is an arbitrary state of stress represented by a point P* on or inside the yield surface, then the difference between the incremental plastic works done by the two stress states on the actual plastic strain rate is

$$dW_p = (\sigma_{ij} - \sigma_{ij}^{*})\, d\varepsilon_{ij}^{p} \tag{3.5}$$

Equation (3.5) represents the scalar product of the vector P*P and PQ. If the yield surface is strictly *convex,* the angle between these vectors is acute and the scalar product is positive. Therefore

$$(\sigma_{ij} - \sigma_{ij}^{*})\, d\varepsilon_{ij}^{p} \geq 0 \tag{3.6}$$

This condition, due to von Mises (1928) and Hill (1948, 1950), is known as the maximum plastic work principle or theorem. It states that the actual work done in a given plastic strain rate (or increment) is greater than the fictitious work done by an arbitrary state of stress not exceeding the yield limit. Alternatively the maximum plastic work principle can be stated as follows: the plastic work done in a given plastic strain rate has a maximum value in the actual state, with respect to varying stress systems satisfying the yield criterion (Hill, 1948, 1950). As will be seen later in this book, the maximum plastic work theorem (3.6) is the basis for a number of important theorems concerning elastic-plastic solids. For example, it can be shown that the stress field in a material that obeys the maximum plastic work principle is always unique.

In short, the maximum plastic work principle is a mathematical statement of the following two important ideas: (a) The yield surface is convex; (b) The plastic strain rate (or increment) is normal to the yield surface.

3.5 STRAIN HARDENING AND PERFECT PLASTICITY

Plastic deformation leads to the hardening of a material and the increase of its elastic limit (i.e. the stress limit under which only elastic deformation occurs). In other words, the yield surface will generally not be fixed in stress space, rather it will expand or contract depending on previous plastic deformation and loading history. Let us for the present consider the case when plastic deformation only changes the size of the yield surface equally in all directions but not its shape (which is known as isotropic hardening). If the yield surface is expanding in size, the material is said to be hardening (i.e. making it more difficult to yield). On the other hand, if the yield surface is contracting in size, then the material is said to be undergoing softening (i.e. making it easier to yield).

The change of the size of the yield surface is often related to some measure or integral of plastic strain rates. The most common measures include the total plastic work per unit volume, the accumulated plastic strain (Hill, 1950), the volumetric plastic strain rate (Schofield and Wroth, 1968; Yu, 1998), or a combination of volumetric and shear plastic strain rates (Wilde, 1977; Yu *et al.*, 2005). The yield surface for a strain-hardening or softening material is also called *the loading surface*. Mathematically, the loading surface, which changes with plastic deformation, may be expressed by

$$f(\sigma_{ij}, \varepsilon_{ij}{}^{p}) = 0 \tag{3.7}$$

where $\varepsilon_{ij}{}^{p}$ denotes the plastic strain tensor.

If the yield surface does not change with stress history (i.e. fixed), the material is known a perfectly plastic solid. This is a special case of strain-hardening materials. For a perfectly plastic material, the behaviour is elastic when the stress state lies inside the yield surface. Plastic strains will occur as long as the stress state lies on or travels along the yield surface. The complete stress conditions for plastic and elastic behaviour may be stated as

$$\textit{Elastic}: \ f(\sigma_{ij}) < 0 \quad \textit{or} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \tag{3.8}$$

$$\textit{Plastic}: \ f(\sigma_{ij}) = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{3.9}$$

The elastic behaviour of a strain-hardening solid is the same as that of a perfectly plastic one. Therefore the conditions for initial yield must be the same. Indeed, the difference between the two concerns only the mechanism for continuing plastic flow, plus the fact that the conditions for current yielding will depend on the plastic history of the material. The complete stress conditions for plastic and elastic behaviour for a strain-hardening material are

$$\textit{Elastic}: \quad f(\sigma_{ij}, \varepsilon_{ij}^p) < 0 \quad or \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \leq 0 \tag{3.10}$$

$$\textit{Plastic}: \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \tag{3.11}$$

Note that in the above conditions, df is evaluated only with respect to the increments in the stress components (that is, with constant plastic strains, see Kachanov, 1974).

For solving boundary value problems involving elastic-plastic behaviour, it is essential to clearly determine what behaviour will result from a further stress increment when the stress state is already on the yield surface. Three possible conditions exist and they are

$$\textit{Unloading}: \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \tag{3.12}$$

$$\textit{Neutral loading}: \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{3.13}$$

$$\textit{Loading}: \quad f(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \tag{3.14}$$

It is commonly assumed that for both unloading and neutral loading, material behaviour is purely elastic. Plastic behaviour occurs only when the loading condition is satisfied.

Based on the loading conditions (3.12)-(3.14), Hill (1950) shows that a general expression for plastic strain rates can be assumed to be

$$d\varepsilon_{ij}^p = G_{ij}\, df \tag{3.15}$$

where G_{ij} is a symmetric tensor, which is supposed to be a function of the stress components and possibly of the previous strain history, but not of the stress rate (or increment). This last assumption is very significant as it means that the ratios of the components of the plastic strain rate are functions of the current stress but not of the stress rate. This can be satisfied by assuming G_{ij} to be of the following form

$$G_{ij} = h \frac{\partial g}{\partial \sigma_{ij}} \tag{3.16}$$

where h and g are scale functions of the stress tensor, and possibly also of the strain history. g is also known as plastic potential. With equation (3.16), the plastic strain rate can be determined by the following equation

$$d\varepsilon_{ij}^{p} = h \frac{\partial g}{\partial \sigma_{ij}} \, df \tag{3.17}$$

which was first used by Melan (1938).

3.6 DRUCKER'S STABILITY POSTULATE

The notations of normality and convexity outlined earlier are just mathematical ideas. In an attempt to provide a missing link between material behaviour and these mathematical ideas, Drucker (1952, 1958) introduced a fundamental stability postulate. In essence, Drucker's stability postulate is a generalization of simple facts which are valid for certain classes of materials, and is not a statement of any thermodynamic principle, as it is often presented (Green and Naghdi, 1965).

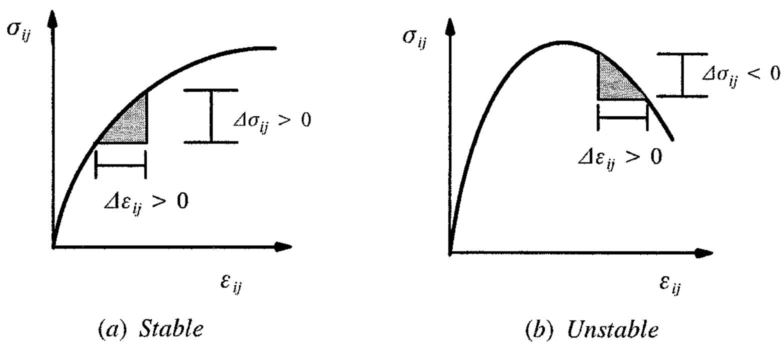


Figure 3.2: Drucker's stability postulate

Figure 3.2 shows two types of typical stress-strain behaviour observed in experiments on real engineering materials. In case (a), the stress increases with increasing strain and the material is actually hardening from the beginning to the end. In other words, an additional loading (i.e., $\Delta\sigma_{ij} > 0$) gives rise to an additional strain (i.e., $\Delta\varepsilon_{ij} > 0$), with the product $\Delta\sigma_{ij}\,\Delta\varepsilon_{ij} > 0$. The additional stress $\Delta\sigma_{ij}$ therefore does positive work as represented by the shaded triangle in the figure. Behaviour of this kind is called *stable*.

In case (b), the deformation curve has a descending branch which follows a strain-hardening section. In the descending section, the strain increases with decreasing stress. In other words, the additional stress does negative work (i.e., $\Delta\sigma_{ij}\,\Delta\varepsilon_{ij} < 0$). Behaviour of this kind is called *unstable*.

In the light of this basic fact, Drucker (1952, 1958) introduced the idea of a *stable plastic material*. This postulate, when applied to an element of elastic-plastic materials in equilibrium under the action of surface loads and body forces, may be stated as follows:

> *Consider an element initially in some state of stress, to which by an external agency an additional set of stresses is slowly applied and slowly removed. Then, during the application of the added stresses and in a cycle of application-and-removal of the added stresses, the work done by the external agency is non-negative.*

If we assume that the existing state of stress (on or inside a loading surface in the stress space) be denoted by $\sigma_{ij}{}^*$, Drucker's stability postulate, as stated above, can be shown to lead to the following two important inequalities (Drucker, 1952, 1960):

$$(\sigma_{ij} - \sigma_{ij}{}^*)\ d\varepsilon_{ij}^{\,p} \geq 0 \tag{3.18}$$

$$d\sigma_{ij}\ d\varepsilon_{ij}^{\,p} \geq 0 \tag{3.19}$$

where (3.18) is in fact the same as the maximum plastic work principle described before in (3.6). It is noted that the equality sign in both (3.18) and (3.19) holds only during neutral loading.

In simpler terms, a material that is stable in Drucker's sense would have the following properties: (a) The yield surface must be convex; (b) The plastic strain rate must be normal to the yield surface (i.e. with an associated flow rule); (c) The rate of strain hardening must be positive or zero (i.e. an additional stress must cause an additional strain); (d) The maximum plastic work principle is valid.

Although Drucker's postulate only covers certain types of real stress-strain behaviour for engineering materials, it does provide a neat way of unifying a whole set of features of plastic stress-strain relations. It must be stressed that while Drucker's postulate implies that the material must obey Hill's maximum plastic work (3.18), the reverse is not true. This is because Drucker's stability postulate also requires a non-decreasing hardening rate, (3.19).

3.7 ISOTROPIC AND KINEMATIC HARDENING

Hardening in the theory of plasticity means that the yield surface changes, in size or location or even in shape, with the loading history (often measured by some form of plastic deformation). When the initial yield condition exists and is identified, the rule of hardening defines its modification during the process of plastic flow.

Most plasticity models currently in use assume that the shape of the yield surface remains unchanged, although it may change in size or location. This restriction is largely based on mathematical convenience, rather than upon any physical principle or experimental evidence. The two most widely used rules of hardening are known as isotropic hardening and kinematic (or anisotropic) hardening.

3.7.1 Isotropic hardening

The rule of isotropic hardening assumes that the yield surface maintains its shape, centre and orientation, but expands or contracts uniformly about the centre of the yield surface.

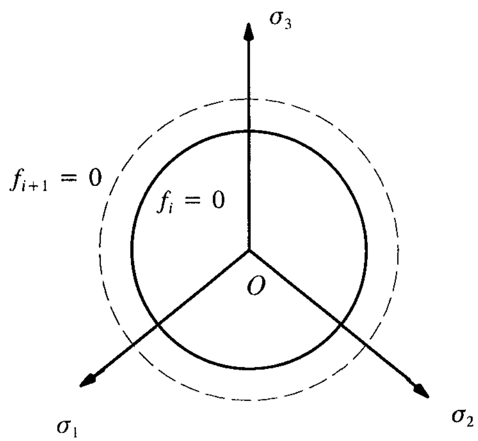


Figure 3.3: Isotropic hardening with uniform expansion of the yield surface

A yield surface with its centre at the origin may be generally described by the following function

$$f = f(\sigma_{ij}) - R(\alpha) = 0 \tag{3.20}$$

where R represents the size of the yield surface, depending on plastic strains through the hardening parameter α. As shown in Hill (1950), the two earliest and most widely used hardening parameters are the accumulated equivalent plastic strain

$$\alpha = \int \sqrt{\frac{2}{3}} (d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p})^{1/2} \tag{3.21}$$

and the plastic work

$$\alpha = \int \sigma_{ij} \, d\varepsilon_{ij}^{p} \tag{3.22}$$

Figure 3.3 shows an example of isotropic hardening where the yield surface is uniformly expanding during the process of plastic flow when a stress increment is applied from step i to $i+1$. The size of the yield surface at any stage of loading is determined as long as an evolution rule defining the relationship between R and α is defined.

3.7.2 Kinematic hardening

The term *kinematic hardening* was introduced by Prager (1955) to construct the first kinematic hardening model. In this first model, it was assumed that during plastic flow, the yield surface translates in the stress space and its shape and size remain unchanged. This is consistent with the Bauschinger effect observed in the uniaxial tension-compression.

Assume that the initial yield surface can be described by

$$f = f(\sigma_{ij} - \alpha_{ij}) - R_0 = 0 \tag{3.23}$$

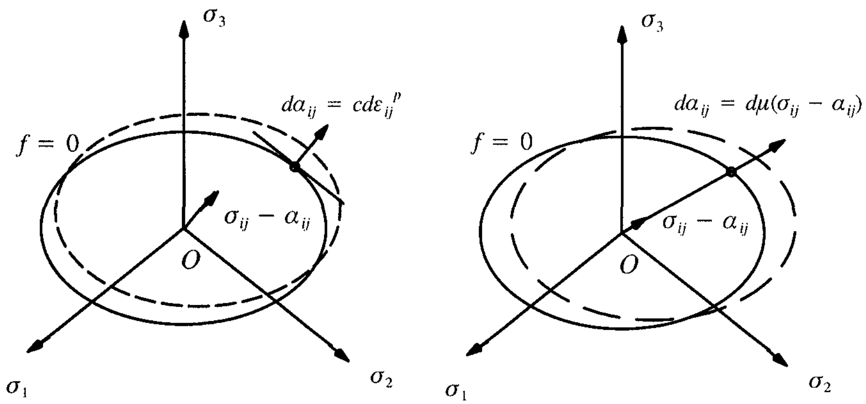
where α_{ij} represents the coordinates of the centre of the yield surface, which is also known as the back stress. R_0 is a material constant representing the size of the original yield surface. It can be seen that as the back stress α_{ij} changes due to plastic flow, the yield surface translates in the stress space while maintaining its initial shape and size.

It is clear now that the formulation of a kinematic hardening model involves assuming an evolution rule of the back stress α_{ij} in terms of ε_{ij}^{p}, σ_{ij} or α_{ij}.

The first simple kinematic hardening model was proposed by Prager (1955). This classical model assumes that the yield surface keeps its original shape and size and moves in the direction of plastic strain rate tensor (see Figure 3.4). Mathematically it can be expressed by the following linear evolution rule

$$d\alpha_{ij} = c \; d\varepsilon_{ij}^{p} \tag{3.24}$$

where c is a material constant.



(a) Prager's hardening (b) Ziegler's hardening

Figure 3.4: Prager's and Ziegler's kinematic hardening

Whilst Prager's model is reasonable for one-dimensional problems, it does not seem to give consistent predictions for two- and three-dimensional cases (Ziegler, 1959). The reason is that the yield function takes different forms for one-, two- and three-dimensional cases. To overcome this limitation, Ziegler (1959) suggested that the yield surface should move in the direction as determined by the vector $\sigma_{ij} - \alpha_{ij}$, see Figure 3.4. Mathematically Ziegler's model can be expressed as follows

$$d\alpha_{ij} = d\mu \; (\sigma_{ij} - \alpha_{ij}) \tag{3.25}$$

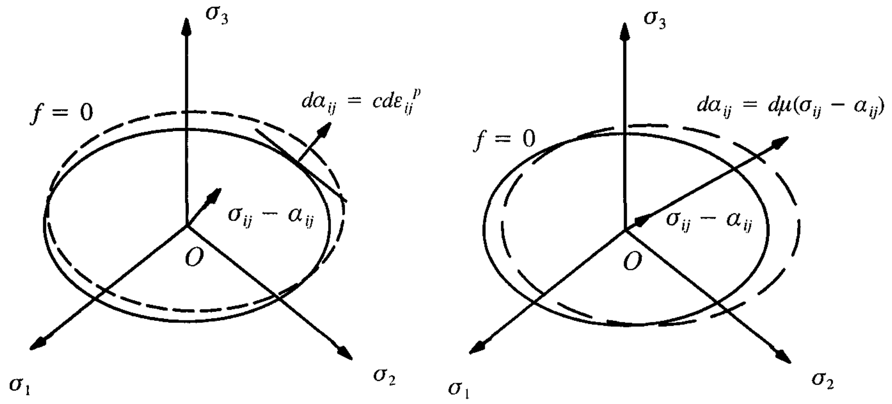
where $d\mu$ is a material constant.

3.7.3 Mixed hardening

The term *mixed hardening* is used to indicate cases when the yield surface not only expands or contracts but also translates in the stress space upon plastic loading (see Figure 3.5). This means that both the centre and size of the yield surface will depend on plastic strain. In this case, the yield function can be expressed by

$$f = f(\sigma_{ij} - a_{ij}) - R(\alpha) = 0 \tag{3.26}$$

where the size of the yield surface can be assumed to be a function of either plastic strain or plastic work., while either Prager's rule (3.24) or Ziegler's rule (3.25) may be used to control the translation of the yield surface upon loading.



(a) Prager's hardening plus isotropic hardening (b) Ziegler's hardening plus isotropic hardening

Figure 3.5: Mixed hardening

3.8 GENERAL STRESS-STRAIN RELATIONS

In order to determine the complete relation between stress and strain for elastic-plastic solids, we still need to assume *consistency* condition (Prager, 1949). For perfectly plastic solids, consistency condition means that the stress state remains on the yield surface. For strain-hardening materials, consistency means that during plastic flow the stress state must remain on the subsequent yield surface (or loading surface). In other words, loading from a plastically deforming state will lead to another plastically deforming state.

3.8.1 Isotropic hardening

For isotropic hardening material, the yield function can be described by

$$f(\sigma_{ij}, \alpha) = 0 \tag{3.27}$$

then Prager's consistency condition requires

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \alpha} d\alpha = 0 \tag{3.28}$$

Since the hardening parameter is a function of plastic strains, so the consistency condition (3.28) can be further written as follows:

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \alpha} \frac{\partial \alpha}{\partial \varepsilon_{ij}{}^p} d\varepsilon_{ij}{}^p = 0 \tag{3.29}$$

For the special case of perfectly plastic solids, the second term of (3.28) will be zero.

The plastic strain rate can be determined from a plastic potential by equation (3.17), which is in fact the same as the plastic flow rule (3.2). This flow rule suggests that once a plastic potential is given, the plastic strain rate will be assumed to be normal to the plastic potential. However the non-negative quantity h or $d\lambda$ needs to be determined in order for the plastic strain rate to be calculated. The consistency condition (3.28) can be used to determined $d\lambda$.

A general procedure for deriving a complete stress-strain relation for perfectly plastic and hardening materials is given below:

(1) To divide the total strain rate (or increment) into elastic and plastic strain rates, namely

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \tag{3.30}$$

(2) Hooke's law is used to link the stress rate with elastic strain rate by elastic stiffness matrix D_{ijkl} as follows

$$d\sigma_{ij} = D_{ijkl}\, d\varepsilon_{kl}^e = D_{ijkl}\, (d\varepsilon_{kl} - d\varepsilon_{kl}^p) \tag{3.31}$$

(3) The general non-associated plastic flow rule is used to express equation (3.31) in the following form

$$d\sigma_{ij} = D_{ijkl}\, (d\varepsilon_{kl} - d\lambda \frac{\partial g}{\partial \sigma_{kl}}) \tag{3.32}$$

(4) By substituting equation (3.32) into the consistency condition (3.29), we obtain

$$d\lambda = \frac{1}{H} \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} d\varepsilon_{kl} \tag{3.33}$$

where H is given by

$$H = \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} - \frac{\partial f}{\partial \alpha} \frac{\partial \alpha}{\partial \varepsilon_{ij}^{p}} \frac{\partial g}{\partial \sigma_{ij}} \tag{3.34}$$

(5) By substituting equation (3.33) into equation (3.32), we obtain a complete relation between a stress rate and a strain rate as follows

$$d\sigma_{ij} = D_{ijkl}^{ep} \, d\varepsilon_{kl} \tag{3.35}$$

where the elastic-plastic stiffness matrix D_{ijkl}^{ep} is defined by

$$D_{ij}^{ep} = D_{ijkl} - \frac{1}{H} D_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} D_{pqkl} \tag{3.36}$$

The above procedure is valid for both strain-hardening and perfectly plastic solids. It is noted that for the case of perfectly plastic solids, the yield surface remains unchanged so that (3.34) takes the following simpler form

$$H = \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \tag{3.37}$$

3.8.2 Kinematic hardening

For kinematic hardening material, the yield function may be expressed as

$$f = f(\sigma_{ij} - \alpha_{ij}) - R_0 = 0 \tag{3.38}$$

where α_{ij} denote the coordinates of the centre of the yield surface, often known as the back stress tensor.

Prager's translation rule

Let us now consider the kinematic hardening law proposed by Prager first, then

$$d\alpha_{ij} = c \, d\varepsilon_{ij}^{p} = c \, d\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{3.39}$$

where g denotes the plastic potential.

With Prager's consistency condition applied to the yield function (3.38), we have

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \alpha_{ij}} d\alpha_{ij} = 0 \tag{3.40}$$

The assumed form of the yield function (3.38) gives

$$\frac{\partial f}{\partial \sigma_{ij}} = -\frac{\partial f}{\partial \alpha_{ij}} \tag{3.41}$$

With equations (3.39) and (3.41), equation (3.40) can be rewritten as

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f}{\partial \sigma_{ij}} c \; d\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{3.42}$$

which gives the plastic multiplier

$$d\lambda = \frac{1}{c} \frac{\dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} = \frac{1}{c} \frac{df}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} \tag{3.43}$$

Then the increments of the back stress tensor and the plastic strains are determined by

$$d\alpha_{ij} = c \; d\varepsilon_{ij}^{p} = c \; d\lambda \frac{\partial g}{\partial \sigma_{ij}} = \frac{df}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} \frac{\partial g}{\partial \sigma_{ij}} \tag{3.44}$$

$$d\varepsilon_{ij}^{p} = \frac{1}{c} \frac{\dfrac{\partial g}{\partial \sigma_{ij}}}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} df \tag{3.45}$$

By using an elastic stress-strain relation, we can determine the elastic strain rate

$$d\varepsilon_{ij}^{e} = C_{ijkl} \; d\sigma_{kl} \tag{3.46}$$

where C_{ijkl} is the elastic compliance matrix. The total strain rate is the sum of the elastic and plastic parts

$$d\varepsilon_{ij} = C_{ijkl} \; d\sigma_{kl} + \frac{1}{c} \frac{\dfrac{\partial g}{\partial \sigma_{ij}}}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} df \tag{3.47}$$

which can be further written as

$$d\varepsilon_{ij} = C^{ep}_{ijkl}\, d\sigma_{kl} \tag{3.48}$$

where the elastic-plastic compliance matrix is

$$C^{ep}_{ijkl} = C_{ijkl} + \frac{1}{c}\frac{\dfrac{\partial g}{\partial \sigma_{ij}}\dfrac{\partial f}{\partial \sigma_{kl}}}{\dfrac{\partial f}{\partial \sigma_{ij}}\dfrac{\partial g}{\partial \sigma_{ij}}} \tag{3.49}$$

It is worth noting that equation (3.48) can be inverted to give

$$d\sigma_{ij} = [C^{ep}_{ijkl}]^{-1}\, d\varepsilon_{kl} = D^{ep}_{ijkl}\, d\varepsilon_{kl} \tag{3.50}$$

Ziegler's translation rule

Ziegler's translation rule is

$$d\alpha_{ij} = d\mu\ (\sigma_{ij} - \alpha_{ij}) \tag{3.51}$$

where $d\mu$ is a constant to be determined.

With Prager's consistency condition applied to the yield function (3.38), we have

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \alpha_{ij})\ d\mu \tag{3.52}$$

which gives

$$d\mu = \frac{\dfrac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\dfrac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \alpha_{ij})} = \frac{df}{\dfrac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \alpha_{ij})} \tag{3.53}$$

The increment of the back stress tensor is therefore given by

$$d\alpha_{ij} = d\mu\ (\sigma_{ij} - \alpha_{ij}) = \frac{df}{\dfrac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \alpha_{ij})}(\sigma_{ij} - \alpha_{ij}) \tag{3.54}$$

It is worth noting that the plastic strain is not involved in the consistency condition with Ziegler's hardening. Therefore the plastic strain cannot be derived from the consistency condition. However it is normally assumed (Melan, 1938) that a plastic modulus exists so that the plastic strain can be derived from a plastic potential following the form of equation (3.17)

$$d\varepsilon_{ij}^{\ p} = \frac{1}{K_p} \frac{\dfrac{\partial g}{\partial \sigma_{ij}}}{\dfrac{\partial f}{\partial \sigma_{ij}} \dfrac{\partial g}{\partial \sigma_{ij}}} df \tag{3.55}$$

where K_p is a material constant known as the plastic modulus which can be determined from the uniaxial compression or tension test. Comparing equation (3.55) with equation (3.45) suggests that the plastic modulus plays the same role as the material constant c in the case with Prager's hardening rule.

3.9 HISTORICAL REMARKS

As rightly pointed out by Koiter (1960), it is often difficult to trace the origin of particular ideas in view of the long and often erratic history of the mathematical theory of plasticity, in particular with regard to the fundamental stress-strain relations. No attempt is made here to give a comprehensive review of the initial history of the plasticity theory. Rather a brief sketch will be given on the major landmarks in the early stage of the development of plastic stress-strain relations. For more detailed discussion, the reader is referred to the reviews given by Hill (1950), Prager (1949, 1955), Prager and Hodge (1951), Koiter (1960), Kachanov (1974) and Martin (1975) among others.

Although the work by Tresca (1864) on the yield criterion of metal is widely regarded as the birth of the classical theory of plasticity, fundamental research on the failure or yielding of soils had been carried out much earlier by Coulomb (1773) and applied by Rankine (1853) to solve earth pressure problems in retaining walls. de Saint-Venant (1870) was the first to develop constitutive relations for perfectly plastic solids. In particular, the coaxial assumption (i.e., requiring coaxiality of the stress tensor and plastic strain tensor) made by de Saint-Venant proved to be a foundation for the classical theory of plasticity with regard to stress-strain relations. A more realistic yield criterion than Tresca's function for metal was proposed by von Mises in 1913. The maximum plastic work principle appears to be first proved by von Mises (1928) and Hill (1948) and then supported by Bishop and Hill (1947) from the behaviour of a single crystal. The development of stress-strain relations for hardening materials in incremental form proceeded very slowly. The first general stress-strain relations for solids with hardening was achieved by Melan (1938) and independently by Prager (1949). To provide a unified theoretical basis for the theory of plasticity, Drucker (1952, 1958) proposed a stability postulate, which includes the principle of maximum plastic work as one of its consequences. Drucker's stability postulate has since been widely used to develop constitutive models for a certain class of plastic solids. In addition, Ziegler (1958) proposed a rather different approach which attempts to bring the theory of plasticity under the scope of Onsager's principle for irreversible thermodynamic processes. This last approach has received more attention in recent years (e.g. Collins and Houlsby, 1997).

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