20

General Elastoplastic Constitutive Models

In the first three parts of this book we focused on the simplest type of plasticity model – an isotropic, perfectly elastoplastic material with an associated flow rule. Such models are sufficiently realistic only under certain conditions and should be used with caution. Now it is time to introduce more advanced concepts.

20.1 HARDENING

Let us recall that a perfectly (or ideally) elastoplastic material subjected to uniaxial loading yields at a constant stress. During plastic flow under general multiaxial loading, the stress state can move along the yield surface, but the surface itself remains unchanged. However, in reality the microstructure of the material changes as plastic flow continues, and this results in a change of the properties observable on the macroscale. Under uniaxial loading, the stress transmitted by a yielding material can increase or decrease. An increase of the yield stress is referred to as *hardening*, and its decrease is called *softening*. Typically, many materials initially harden and later soften. For convenience, however, we will sometimes use the term 'hardening' in a broader sense, meaning yield stress changes of any sign, negative hardening being equivalent to softening.

During hardening (in the broad sense), the elastic domain undergoes a certain evolution. The elastic domain of a virgin material is bounded by the initial yield surface, also called the elastic limit envelope. Due to microstructural changes in the material induced by plastic flow, the elastic domain changes its size or position, or both. Its boundary at an intermediate state is usually called a *loading surface*. Some models work with a maximum possible elastic domain, which is bounded by the *strength envelope* (failure surface), representing the largest possible resistance of the material.

20.1.1 Isotropic Hardening

In order to describe the evolution of the loading surface, we need one or several new parameters that characterize the effect of hardening. The simplest approach, introduced by Odqvist (1933a), deals with a one-parameter family of loading surfaces that are all similar and affine with regard to the origin (Figure 20.1). This is called *isotropic hardening*. The loading surfaces can be derived from the same basic form of

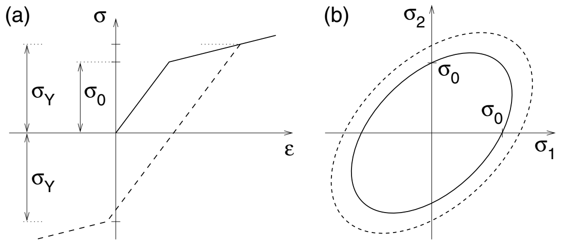

Figure 20.1 Isotropic hardening: a) uniaxial stress-strain diagram, b) evolution of the yield surface in the biaxial stress plane

the yield function, in which different yield stress values are used. Any yield criterion originally stated in the form

$$F(\boldsymbol{\sigma}) - \sigma_0 = 0 \tag{20.1}$$

where σ_0 is a constant yield stress, can be reformulated as

$$F(\boldsymbol{\sigma}) - \sigma_Y = 0 \tag{20.2}$$

where σ_Y is a new variable, the current yield stress, initially equal to the material parameter σ_0.

The evolution of the yield stress during plastic flow must be described by an additional equation, the *hardening law*. Simple examples of hardening in one dimension have already been presented in Chapter 1. Hardening laws under uniaxial monotonic loading can be postulated as an explicit dependence of the yield stress on the plastic strain,

$$\sigma_Y = h(\varepsilon_{11}^p) \tag{20.3}$$

which is called the *strain hardening*. Here, ε_{11}^p is the plastic part of the normal strain ε_{11} in the direction of applied stress σ_{11}. The function h is easily extracted from the monotonic uniaxial stress-strain curve. Its derivative,

$$h'(\varepsilon_{11}^p) \equiv H(\varepsilon_{11}^p) \tag{20.4}$$

is called the *plastic modulus*. In the simplest case, the hardening law is linear, $\sigma_Y = \sigma_0 + H\varepsilon_{11}^p$, and the plastic modulus is constant, $H(\varepsilon_{11}^p) = H$. For a positive plastic modulus we get true hardening in the sense of an increase of the yield stress; $H = 0$ corresponds to perfect plasticity; and for $H < 0$ the material exhibits softening. A negative plastic modulus is often called the softening modulus.

In a general multiaxial setting, the plastic strain $\boldsymbol{\varepsilon}_p$ is a second-order tensor, and the one-dimensional hardening law (20.3) can be extended to this case if we introduce a scalar measure that reflects the amount of changes in the material microstructure. The simplest choice might seem to be the norm of the plastic strain tensor, $\|\boldsymbol{\varepsilon}_p\| = \sqrt{\boldsymbol{\varepsilon}_p : \boldsymbol{\varepsilon}_p}$, but this variable does not always increase during plastic flow. The reason is that the plastic strain rate can in general have any direction, which causes that the norm of $\boldsymbol{\varepsilon}_p$ can decrease even when the plastic flow continues. Therefore it makes more sense to characterize strain hardening by the *cumulative plastic strain* (also called the effective plastic strain or the equivalent plastic strain), $\bar{\varepsilon}_p$, defined by the rate equation

$$\dot{\bar{\varepsilon}}_p = \sqrt{\frac{2}{3}}\,\|\dot{\boldsymbol{\varepsilon}}_p\| = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p} \tag{20.5}$$

This was first proposed, without the scaling factor $\sqrt{2/3}$, by Odqvist (1993a). The scaling factor is chosen such that, under monotonic uniaxial loading, $\dot{\bar{\varepsilon}}_p$ would coincide with the component $\dot{\varepsilon}^p_{11}$ of $\dot{\boldsymbol{\varepsilon}}_p$, provided that the plastic flow is isochoric (purely deviatoric), i.e. no plastic change of volume takes place. Definition (20.5) can be integrated to yield

$$\bar{\varepsilon}_p(t) = \sqrt{\frac{2}{3}} \int_0^t \|\dot{\boldsymbol{\varepsilon}}_p(\tau)\| \, \mathrm{d}\tau \tag{20.6}$$

where the time-like variable t can be any monotonically increasing parameter controling the loading process.

An alternative to the strain hardening hypothesis (i.e. to the assumption that the yield stress depends on the cumulative plastic strain) is the *work hardening* hypothesis (Taylor and Quinney, 1931), stating that the yield stress depends on the plastic work,

$$W_p(t) = \int_0^t \boldsymbol{\sigma}(\tau) : \dot{\boldsymbol{\varepsilon}}_p(\tau) \, \mathrm{d}\tau \tag{20.7}$$

The choice between strain and work hardening depends on multiaxial experiments. If only uniaxial test results are available, there is no way to decide which hypothesis is better. Both are widely used, and even in many multiaxial stress situations they lead to similar results. For associated J_2-plasticity they are completely equivalent (after an appropriate transformation of the hardening function h).

To cover both cases by a unique description, we will deal with the notion of a hardening variable, denoted as κ, which can be either the cumulative plastic strain (for strain hardening) or the plastic work (for work hardening). It is convenient, albeit not necessary, to scale down the plastic work such that it has the dimension of strain and that, under uniaxial tension, it corresponds to ε^p_{11}. This can be achieved by defining the work hardening variable by the rate equation

$$\dot{\kappa} = \frac{1}{h(\kappa)} \, \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p \tag{20.8}$$

Equation (20.3) is then written as

$$\sigma_Y = h(\kappa) \tag{20.9}$$

Note that, during plastic flow under uniaxial tension, $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p = \sigma_{11}\dot{\varepsilon}^p_{11} = \sigma_Y \dot{\varepsilon}^p_{11}$, and so (20.8) combined with (20.9) gives $\dot{\kappa} = \dot{\varepsilon}^p_{11}$. Consequently, the strain hardening variable and the work hardening variable are identical under uniaxial stress conditions, and the same hardening function h can then be used for either type of hardening.

Let us now explore the effect of hardening on the elastoplastic stiffness tensor and on the uniqueness of the model response. Here we deal only with uniqueness in the local sense, i.e. we check whether the response of a material point to any prescribed strain evolution is unique. Compared to the perfectly plastic model analyzed in Chapter 15, we now have two additional variables – the current yield stress, σ_Y, and the hardening variable, κ. The corresponding additional equations are the two components of the hardening law – the dependence of the current yield stress on the hardening variable, and the definition of the hardening variable. Since (20.9) explicitly relates the current yield stress to the hardening variable, it is sufficient to introduce only the hardening variable as a primary unknown. The yield function is reformulated as

$$f(\boldsymbol{\sigma}, \kappa) = F(\boldsymbol{\sigma}) - h(\kappa) \tag{20.10}$$

and its time derivative is

$$\dot{f} = \frac{\partial F}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} - h' \, \dot{\kappa} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} - H\dot{\kappa} \tag{20.11}$$

Note that the gradients of f and F with respect to $\boldsymbol{\sigma}$ are identical, $\partial F/\partial\boldsymbol{\sigma} = \partial f/\partial\boldsymbol{\sigma}$, and so we can use for $\partial F/\partial\boldsymbol{\sigma}$ the symbol $\boldsymbol{f}_\sigma$, introduced in Chapter 15 for the gradient $\partial f/\partial\boldsymbol{\sigma}$. Of course, $\boldsymbol{f}_\sigma$ depends on the current value of $\boldsymbol{\sigma}$, and H may depend on the current value of κ (if the hardening law is nonlinear) but, for the sake of simplicity, we do not mark these dependencies explicitly.

Some important criteria, e.g. those due to von Mises, or Drucker and Prager, are written in their most natural form in terms of an alternative yield stress measure, e.g. the yield stress in shear, τ_0. For hardening materials, such criteria can be written in the general form

$$f(\boldsymbol{\sigma}, \kappa) = F(\boldsymbol{\sigma}) - \bar{h}(\kappa) \tag{20.12}$$

where

$$\bar{h}(\kappa) = \frac{\sigma_0^{ch}}{\sigma_0} h(\kappa) \tag{20.13}$$

is the hardening function rescaled by the ratio of the characteristic yield stress, σ_0^{ch}, to the yield stress in uniaxial tension, σ_0. For example, a criterion originally written as $F(\boldsymbol{\sigma}) - \tau_0 = 0$ will be reformulated as $F(\boldsymbol{\sigma}) - \bar{h}(\kappa) = 0$, where $\bar{h}(\kappa) = (\tau_0/\sigma_0)h(\kappa)$. The time derivative of the yield function is then

$$\dot{f} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} - \bar{H}\dot{\kappa} \tag{20.14}$$

where

$$\bar{H} = \bar{h}' = \frac{\sigma_0^{ch}}{\sigma_0} h' = \frac{\sigma_0^{ch}}{\sigma_0} H \tag{20.15}$$

is the rescaled plastic modulus.

For strain hardening as well as work hardening, the rate of the hardening variable, $\dot{\kappa}$, is proportional to the rate of the plastic multiplier, $\dot{\lambda}$, with a proportionality factor dependent on the current stress. Indeed, using the associated flow rule (15.28), for strain hardening we obtain

$$\dot{\kappa} = \sqrt{\frac{2}{3}} \, \|\dot{\boldsymbol{\varepsilon}}_p\| = \sqrt{\frac{2}{3}} \, \|\boldsymbol{f}_\sigma\|\dot{\lambda} \tag{20.16}$$

and for work hardening

$$\dot{\kappa} = \frac{1}{h(\kappa)} \, \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p = \frac{\boldsymbol{\sigma} : \boldsymbol{f}_\sigma}{h(\kappa)} \, \dot{\lambda} \tag{20.17}$$

Both cases can be covered by a general relation

$$\dot{\kappa} = k\dot{\lambda} \tag{20.18}$$

where k is a scalar factor;

$$k = \sqrt{\frac{2}{3}} \, \|\boldsymbol{f}_\sigma\| \tag{20.19}$$

for strain hardening and

$$k = \frac{\boldsymbol{\sigma} : \boldsymbol{f}_\sigma}{h(\kappa)} \tag{20.20}$$

for work hardening. Recall that if the elastic domain is convex and contains the origin (the initial stress-free state), then $\boldsymbol{\sigma} : \boldsymbol{f}_\sigma > 0$. The factor k is, therefore, always positive.

Example 20.1: Evaluate the factor k from (20.18) for associated J_2-plasticity.

Solution: Hardening J_2-plasticity is described by the yield function

$$f(\boldsymbol{\sigma}, \kappa) = \sqrt{J_2(\boldsymbol{\sigma})} - \bar{h}(\kappa) \tag{20.21}$$

where $\bar{h}(\kappa) = (\tau_0/\sigma_0)h(\kappa) = h(\kappa)/\sqrt{3}$. The gradient of the yield function with respect to the stress tensor is $\boldsymbol{f}_\sigma = \boldsymbol{s}/(2\sqrt{J_2})$. For strain hardening we obtain

$$k = \sqrt{\frac{2}{3}}\|\boldsymbol{f}_\sigma\| = \frac{1}{\sqrt{6J_2}}\|\boldsymbol{s}\| = \frac{1}{\sqrt{6J_2}}\sqrt{2J_2} = \frac{1}{\sqrt{3}} \tag{20.22}$$

while for work hardening we have

$$k = \frac{\boldsymbol{\sigma} : \boldsymbol{f}_\sigma}{h(\kappa)} = \frac{\boldsymbol{s} : \boldsymbol{s}}{2\sqrt{J_2}h(\kappa)} = \frac{\sqrt{J_2}}{h(\kappa)} = \frac{\bar{h}(\kappa)}{h(\kappa)} = \frac{\tau_0}{\sigma_0} = \frac{1}{\sqrt{3}} \tag{20.23}$$

because, during plastic flow, $\sqrt{J_2} = \bar{h}(\kappa)$. We can see that, in this specific case, the factor k is constant and has the same value for strain hardening as for work hardening. □

In summary, isotropically hardening elastoplasticity is described by the elastic stress-strain law,

$$\boldsymbol{\sigma} = \boldsymbol{D}_e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) \tag{20.24}$$

the associated flow rule,

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\boldsymbol{f}_\sigma \tag{20.25}$$

the evolution law for the hardening variable,

$$\dot{\kappa} = \dot{\lambda}k \tag{20.26}$$

and by the loading-unloading conditions,

$$f \le 0, \quad \dot{\lambda} \ge 0, \quad f\dot{\lambda} = 0 \tag{20.27}$$

in which f is a function of $\boldsymbol{\sigma}$ and κ having the form (20.10), and k is a function of $\boldsymbol{\sigma}$ and κ given by (20.19) or (20.20). Our aim is to find the rates $\dot{\boldsymbol{\sigma}}$, $\dot{\boldsymbol{\varepsilon}}_p$, $\dot{\lambda}$ and $\dot{\kappa}$ for a given strain rate $\dot{\boldsymbol{\varepsilon}}$, provided that the current values of all the variables are known.

Following the same line of reasoning as in Chapter 15, we analyze the cases of plastic loading and of unloading from a plastic state separately. The case of an elastic process inside the elastic domain is trivial.

1. During plastic flow, the stress state must remain on the yield surface, and the consistency condition
$$\dot{f} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} - \bar{H}\dot{\kappa} = 0 \tag{20.28}$$
must be satisfied. Unlike perfect plasticity, the yield surface does not remain stationary but evolves as the plastic flow continues. This is reflected by the term $-\bar{H}\dot{\kappa}$ in (20.28). Now we substitute (20.24)–(20.26) into the consistency condition (20.28) so as to eliminate all the unknowns except $\dot{\lambda}$. The first term, $\boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}}$, can be handled in exactly the same manner as in Chapter 15; cf. equation (15.40). In the second term, we simply substitute (20.26) for $\dot{\kappa}$. Thus the consistency condition is transformed into the equation
$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma) - \bar{H}k\dot{\lambda} = 0 \tag{20.29}$$
which contains a single unknown, $\dot{\lambda}$. The solution
$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k} \tag{20.30}$$
is admissible if the resulting value of $\dot{\lambda}$ is nonnegative. Back-substitution into the flow rule (20.25) and the elastic law (20.24) then leads to the rate form (or tangential form) of the stress-strain equations,
$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_{ep} : \dot{\boldsymbol{\varepsilon}} \tag{20.31}$$
where
$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{\boldsymbol{D}_e : \boldsymbol{f}_\sigma \otimes \boldsymbol{f}_\sigma : \boldsymbol{D}_e}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k} \tag{20.32}$$
is the elastoplastic material stiffness tensor. The only difference compared to (15.47) is the presence of the term $\bar{H}k$ in the denominator. For $\bar{H} = 0$ we recover formula (15.47) valid for a perfectly elastoplastic material.
2. During elastic unloading from a plastic state we have $\dot{\lambda} = 0$, and so $\dot{\boldsymbol{\varepsilon}}_p = \boldsymbol{0}$, $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}$, and $\dot{\kappa} = 0$. However, the solution is plastically admissible only if $\dot{f} \le 0$. Substituting into (20.14), we obtain
$$\dot{f} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} - \bar{H}\dot{\kappa} = \boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} \tag{20.33}$$

Having derived the solutions valid for plastic loading and for unloading from a plastic state, we can address the issue of uniqueness. The former solution is admissible if the rate of the plastic multiplier is nonnegative, i.e. if
$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k} \ge 0 \tag{20.34}$$
The latter solution is admissible if the time derivative of the yield function is nonpositive, i.e. if
$$\dot{f} = \boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} \le 0 \tag{20.35}$$
The denominator in (20.34) depends only on the current state and can be evaluated independently of the prescribed strain rate. If this denominator is positive, then condition (20.34) is equivalent with
$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} \ge 0 \tag{20.36}$$

and thus is complementary to (20.35). This means that, for any prescribed strain rate, exactly one of the conditions is satisfied and the solution is unique. Both conditions are simultaneously satisfied only in the case of neutral loading when $\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} = 0$, but then both solutions are identical and uniqueness is not violated.

On the other hand, if the denominator in (20.34) is negative, conditions (20.34) and (20.35) are equivalent and the problem either has two different solutions or no solution at all, depending on whether the prescribed strain rate satisfies or violates (20.35). This is clearly an undesirable situation. So we can conclude that the response to any prescribed strain evolution is unique if

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k > 0 \tag{20.37}$$

for any state of the material. Note that the first term, $\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma$, is always positive due to the positive definiteness of $\boldsymbol{D}_e$. The factor k is also always positive. If the plastic modulus is positive or zero, the condition of uniqueness (20.37) is satisfied for any possible state of the material. This means that the response of hardening (in the narrow sense, $\bar{H} > 0$) or perfectly plastic ($\bar{H} = 0$) materials with an associated flow is always locally unique, i.e. any prescribed strain history generates a unique response in terms of stress, plastic strain, etc. However, the condition of uniqueness might be satisfied by softening materials as well, as is demonstrated by the following example.

Example 20.2: Consider an isotropically hardening or softening von Mises material with an associated flow rule. Express the condition of uniqueness in terms of the plastic modulus.

Solution: The yield function for von Mises material is given by

$$f(\boldsymbol{\sigma}, \kappa) = \sqrt{J_2(\boldsymbol{\sigma})} - \bar{h}(\kappa) \tag{20.38}$$

According to (20.37), uniqueness is guaranteed if $\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k > 0$. Substituting the gradient $\boldsymbol{f}_\sigma = \partial f/\partial\boldsymbol{\sigma} = \boldsymbol{s}/(2\sqrt{J_2})$ into the first term, we obtain

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma = \frac{1}{4J_2}\boldsymbol{s} : \boldsymbol{D}_e : \boldsymbol{s} = \frac{1}{4J_2}\boldsymbol{s} : (2G\boldsymbol{s}) = \frac{G}{2J_2}\boldsymbol{s} : \boldsymbol{s} = G \tag{20.39}$$

In Example 20.1 we have shown that, for J_2-plasticity, $k = 1/\sqrt{3}$, independently of the type of hardening hypothesis. The condition of uniqueness thus reads

$$G + \frac{1}{\sqrt{3}}\bar{H} > 0 \tag{20.40}$$

i.e., $\bar{H} > -\sqrt{3}G$. So local uniqueness is preserved even for a softening material provided that the magnitude of the softening modulus does not exceed a certain critical value related to the elastic shear modulus. In terms of the standard softening modulus, $H = (\sigma_0/\tau_0)\bar{H} = \sqrt{3}\bar{H}$, the condition of uniqueness would read $H > -3G$. □

20.1.2 Linear Kinematic Hardening

So far we have considered only isotropic hardening, characterized by a single parameter. It appears, however, that more complicated hardening rules are necessary,

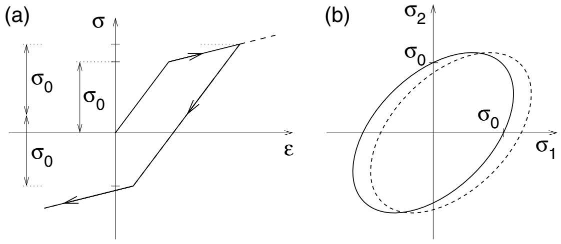

Figure 20.2 Kinematic hardening: a) uniaxial stress-strain diagram, b) evolution of the yield surface in the biaxial stress plane

especially for the case of unloading and cyclic loading. As an alternative hardening rule, the current loading surface is assumed not to expand but to move as a rigid body within the stress space (Figure 20.2(b)); this is known as *kinematic hardening*. The use of kinematic hardening is, for example, necessary to model the so-called *Bauschinger effect* (Bauschinger, 1881). This effect is often observed in metals subjected to cyclic loading. Even if the magnitudes of the yield stress in tension and in compression are initially the same, this is no longer the case when the material is preloaded into the plastic range and then unloaded. For example, after previous yielding in tension, yielding in compression may start at a stress level lower than the initial yield stress (Figure 20.2(a)).

The hardening behavior rule of most materials appears to be a combination of the isotropic and kinematic type of hardening, sometimes accompanied by a change of shape of the yield surface (see the discussion of the vertex effect in Section 25.4).

Kinematic hardening leads to a translation of the loading surface, i.e. to a shift of the origin of the initial yield surface. If the initial yield surface is described by a yield function of the form

$$f(\boldsymbol{\sigma}) = F(\boldsymbol{\sigma}) - \sigma_0 \tag{20.41}$$

the shifted surface is obviously described by

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b) = F(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b) - \sigma_0 \tag{20.42}$$

where $\boldsymbol{\sigma}_b$ is the so-called *back stress* that represents the center of the shifted elastic domain and plays the role of a tensorial hardening variable. Now we need a kinematic hardening law that governs the evolution of the back stress. Melan (1938b) proposed a law of the form

$$\dot{\boldsymbol{\sigma}}_b = \bar{H}_K \, \dot{\boldsymbol{\varepsilon}}_p \tag{20.43}$$

according to which the rate of the back stress is proportional to the plastic strain rate. The proportionality factor $\bar{H}_K$ is directly related to the plastic modulus; see Problem 20.5. The linear hardening law (20.43) is often credited to Prager (1955, 1956); we will call it the *Melan–Prager hardening rule*. Generalizations are due to Backhaus (1968), who considered $\bar{H}_K$ as a variable dependent on the cumulative plastic strain, and Lehmann (1972), who replaced it by a tensor $\bar{\boldsymbol{H}}_K$.

Following the usual procedure, we construct the consistency condition

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\sigma}_b} : \dot{\boldsymbol{\sigma}}_b = 0 \tag{20.44}$$

and then use the elastic stress-strain law, flow rule and hardening law so as to eliminate all the unknowns except the rate of the plastic multiplier. Denoting again $\partial f/\partial\boldsymbol{\sigma} = \partial F/\partial\boldsymbol{\sigma} = \boldsymbol{f}_\sigma$, and realizing that $\partial f/\partial\boldsymbol{\sigma}_b = -\partial F/\partial\boldsymbol{\sigma} = -\boldsymbol{f}_\sigma$, we can write the resulting equation as

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma) - \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma \bar{H}_K \dot{\lambda} = 0 \tag{20.45}$$

from which

$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma} \tag{20.46}$$

This result is very similar to that obtained for isotropic hardening, with the only difference that the term $\bar{H}k$ is replaced by the term $\bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma$; cf. (20.30). It is now easy to obtain the elastoplastic stiffness tensor

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{\boldsymbol{D}_e : \boldsymbol{f}_\sigma \otimes \boldsymbol{f}_\sigma : \boldsymbol{D}_e}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma} \tag{20.47}$$

and the condition of local uniqueness

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma > 0 \tag{20.48}$$

For nonnegative values of the plastic modulus, this condition is always satisfied.

Negative values of $\bar{H}_K$ would lead to softening. A model with purely kinematic softening does not make physical sense but a model with mixed isotropic and kinematic softening might be useful, e.g. for materials in which tensile loading induces a degradation of both tensile and compressive strength, but the degradation in tension is faster than in compression.

Example 20.3: Derive the specific form of the elastoplastic stiffness matrix for associated J_2-plasticity with kinematic hardening according to the Melan–Prager rule.

Solution: Partial differentiation of the yield function $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b) = \sqrt{J_2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)} - \tau_0$ with respect to the stress tensor $\boldsymbol{\sigma}$ at points where $f(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b) = 0$ gives $\boldsymbol{f}_\sigma = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)_{\text{dev}}/2\tau_0$, where $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)_{\text{dev}}$ denotes the deviatoric part of $\boldsymbol{\sigma} - \boldsymbol{\sigma}_b$. Realizing that the rate of the back stress is proportional to the rate of plastic strain, which is purely deviatoric, we conclude that the back stress $\boldsymbol{\sigma}_b$ is also purely deviatoric, and so $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)_{\text{dev}} = \boldsymbol{s} - \boldsymbol{\sigma}_b$. Substituting this into (20.47) and using the relations $\boldsymbol{D}_e : (\boldsymbol{s} - \boldsymbol{\sigma}_b) = 2G(\boldsymbol{s} - \boldsymbol{\sigma}_b)$ and $(\boldsymbol{s} - \boldsymbol{\sigma}_b) : (\boldsymbol{s} - \boldsymbol{\sigma}_b) = 2J_2(\boldsymbol{s} - \boldsymbol{\sigma}_b) = 2\tau_0^2$, we obtain

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{2G^2}{\tau_0^2(2G + \bar{H}_K)} (\boldsymbol{s} - \boldsymbol{\sigma}_b) \otimes (\boldsymbol{s} - \boldsymbol{\sigma}_b) \tag{20.49}$$

□

Ziegler (1959) proposed a modification of the Melan–Prager kinematic hardening rule. He observed that if the original equation (20.43) is reduced to a subspace of the stress space (e.g. to the subspace corresponding to the plane stress), the yield surface does not always move in the direction of its normal at the current stress point. In some cases, e.g. for the Tresca condition in the σ_x–τ_{xy} space, the yield surface even deforms during kinematic hardening according to the Melan–Prager rule. Of course, the complete yield surface in the nine-dimensional stress space does not deform but, as it moves, the shape of its intersection with a given stress subspace can change. This might be annoying and even confusing.

Ziegler also observed that the von Mises yield surface following the Melan–Prager rule always moves in the direction of the vector connecting its center with the current stress point, and that this property holds in any subspace of the stress space. Therefore, he suggested a modified kinematic hardening law

$$\dot{\boldsymbol{\sigma}}_b = \dot{\mu}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b) \tag{20.50}$$

where $\dot{\mu}$ is a new multiplier, to be determined from a suitable additional condition. Equation (20.50) only defines the direction in which the yield surface moves, but the rate at which this happens remains unspecified. The consistency condition

$$\boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} - \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}}_b = 0 \tag{20.51}$$

provides (after the usual substitution from the elastic stress-strain law, flow rule, and hardening law) only one equation for two unknowns – the plastic multiplier, $\dot{\lambda}$, and the multiplier from the hardening law, $\dot{\mu}$:

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma) - \dot{\mu}\boldsymbol{f}_\sigma : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_b) = 0 \tag{20.52}$$

Note that no parameter playing the role of a plastic modulus has been introduced yet, and the relationship between the stress rate and the plastic strain rate remains unspecified. It therefore makes sense to postulate a condition that the projection of the stress rate onto the direction of the normal to the yield surface is proportional to the projection of the plastic strain rate. This supplementary condition,

$$\dot{\boldsymbol{\sigma}} : \boldsymbol{f}_\sigma = \bar{H}_K \, \dot{\boldsymbol{\varepsilon}}_p : \boldsymbol{f}_\sigma \tag{20.53}$$

is in fact satisfied by the model with Melan–Prager hardening as well, which is verified by substituting the hardening law (20.43) into the consistency condition (20.51). Thus, the parameter $\bar{H}_K$ has the same meaning as in the Melan–Prager hardening rule.

Again, using the elastic stress-strain law and the flow rule, we obtain an equation with a single unknown,

$$\boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma) = \bar{H}_K \dot{\lambda} \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma \tag{20.54}$$

which is easily solved to give

$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma} \tag{20.55}$$

Upon substitution into (20.52), we finally obtain

$$\dot{\mu} = \bar{H}_K \frac{\boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma}{\boldsymbol{f}_\sigma : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)} \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma} \tag{20.56}$$

Thus, for a prescribed strain rate, the rate of the back stress can be evaluated from (20.50) with $\dot{\mu}$ given by (20.56).

For the von Mises condition, Ziegler's rule leads to exactly the same results as the Melan–Prager rule. However, for other yield conditions the response of the model in general depends on the specific form of the hardening rule. According to Ziegler's rule, the yield surface moves in the direction connecting its current center with the current stress point, and this remains true in any reduced stress space.

20.1.3 Mixed Hardening

Mixed hardening models combine isotropic and kinematic hardening, which results into a modification of the loading surface by simultaneous translation and expansion (or contraction). It is easy to design a yield function

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b, \kappa) = F(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b) - \bar{h}(\kappa) \tag{20.57}$$

that covers functions (20.12) and (20.42) as special cases. Derivation of the elastoplastic stiffness tensor

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{\boldsymbol{D}_e : \boldsymbol{f}_\sigma \otimes \boldsymbol{f}_\sigma : \boldsymbol{D}_e}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}_K \boldsymbol{f}_\sigma : \boldsymbol{f}_\sigma + \bar{H} k} \tag{20.58}$$

is left to the reader as an easy excercise. Generalization of the conditions of local uniqueness (20.37) and (20.48) is also straightforward.

Example 20.4: Develop an associated J_2-plasticity model combining linear isotropic and linear kinematic hardening.

Solution: The yield condition is given by

$$\sqrt{J_2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_b)} - \bar{h}(\kappa) = 0 \tag{20.59}$$

For convenience, we rewrite it in the equivalent form

$$f(\boldsymbol{\sigma}, \boldsymbol{\sigma}_b, \kappa) \equiv \frac{1}{2}(\boldsymbol{s} - \boldsymbol{\sigma}_b) : (\boldsymbol{s} - \boldsymbol{\sigma}_b) - \bar{h}^2(\kappa) = 0 \tag{20.60}$$

The gradient of the yield function (20.60) with respect to the stress tensor is $\boldsymbol{f}_\sigma = \boldsymbol{s} - \boldsymbol{\sigma}_b$, and so the associated flow rule reads

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}(\boldsymbol{s} - \boldsymbol{\sigma}_b) \tag{20.61}$$

Linear isotropic hardening can be described by

$$\bar{h}(\kappa) = \frac{\tau_0}{\sigma_0} h(\kappa) = \frac{\sqrt{3}}{3}(\sigma_0 + H_I \kappa) \tag{20.62}$$

where H_I is the (constant) isotropic plastic modulus, and $\kappa \equiv \bar{\varepsilon}_p$ is the cumulative plastic strain defined by (20.6). Linear kinematic hardening according to the Melan–Prager rule is characterized by a linear dependence of the back stress on the plastic strain,

$$\boldsymbol{\sigma}_b = \bar{H}_K \boldsymbol{\varepsilon}_p \tag{20.63}$$

□

20.1.4 Nonlinear Kinematic Hardening

Realistic modeling of engineering materials often requires nonlinear hardening laws. For isotropic hardening, nonlinearity can be easily incorporated through the function $h(\kappa)$, but for kinematic hardening this is not so simple. At first, one might think that it is sufficient to replace the constant $\bar{H}_K$ in (20.43) by a function of the cumulative plastic strain. However, this would seldom lead to realistic shapes of the stress-strain diagrams under cyclic loading.

A suitable form of the nonlinear kinematic hardening law was proposed by Armstrong and Frederick (1966). They enriched the linear Melan–Prager rule (20.43) by a term proportional to the current back stress multiplied by the norm of the plastic strain rate. According to the *Armstrong–Frederick rule*, the evolution of the back stress is governed by the differential equation

$$\dot{\boldsymbol{\sigma}}_b = \bar{H}_K \dot{\boldsymbol{\varepsilon}}_p - \gamma\sqrt{\frac{2}{3}}\,\|\dot{\boldsymbol{\varepsilon}}_p\|\,\boldsymbol{\sigma}_b \tag{20.64}$$

where $\bar{H}_K$ and γ are constant material parameters. At the onset of yielding, the back stress is still zero and (20.64) gives the same response as the linear hardening law (20.43). As the back stress develops, the additional term becomes activated and slows down the rate at which the back stress grows (i.e. reduces the tangent plastic modulus), as long as the loading remains monotonic. Upon a load reversal, the back stress and its rate have opposite directions (in the sense that their scalar product is negative) and the additional term increases the plastic modulus. The following example shows that this interplay between the value of the back stress and its rate leads to reasonable shapes of the cyclic stress-strain diagrams.

Example 20.5: Plot the uniaxial cyclic stress-strain diagram for an elastoplastic model with a pressure-independent yield condition, associated flow rule and nonlinear kinematic hardening of the Armstrong–Frederick type. The material is characterized by Young's modulus $E = 200$ GPa, initial uniaxial yield stress $\sigma_0 = 100$ MPa, initial plastic modulus $H_K = 150$ GPa, and parameter $\gamma = 3000$.

Solution: A flow rule associated with a pressure-insensitive yield condition results into purely deviatoric plastic flow. Therefore, the back stress will be a purely deviatoric tensor. Under uniaxial stress σ_x, the largest principal deviatoric stress, acting in the x-direction, is $s_x = \frac{2}{3}\sigma_x$. The corresponding principal value of the back stress will be conveniently denoted as $\sigma_{bx} = \frac{2}{3}\sigma_b$, because this notation leads to the uniaxial yield condition in the form $|\sigma_x - \sigma_b| = \sigma_0$.

If ε_{px} denotes the plastic strain in the x-direction, it follows from axial symmetry and from the condition of plastic incompressibility that the other principal plastic strains are $\varepsilon_{py} = \varepsilon_{pz} = -\frac{1}{2}\varepsilon_{px}$. The 'physical' plastic modulus H_K gives the slope of the curve relating the uniaxial stress σ_x to the plastic strain ε_{px}. To obtain the parameter $\bar{H}_K$, it must be multiplied by a factor 2/3; see Problem 20.5. The hardening law (20.64) written for the normal components in the x-direction and scaled by the factor 3/2 gives

$$\dot{\sigma}_b = H_K \dot{\varepsilon}_{px} - \gamma|\dot{\varepsilon}_{px}|\sigma_b \tag{20.65}$$

During each interval in which the plastic strain rate keeps the same sign, the particular solution of this differential equation satisfying the initial condition $\sigma_b(\varepsilon_{p0}) = \sigma_{b0}$ is given by

$$\sigma_b(\varepsilon_{px}) = \frac{H_K}{\gamma}\operatorname{sgn}\dot{\varepsilon}_{px} + \left(\sigma_{b0} - \frac{H_K}{\gamma}\operatorname{sgn}\dot{\varepsilon}_{px}\right)\exp\left[-\gamma(\varepsilon_{px} - \varepsilon_{p0})\operatorname{sgn}\dot{\varepsilon}_{px}\right] \tag{20.66}$$

During plastic flow we have $\sigma_x - \sigma_b = \sigma_0 \operatorname{sgn}\dot{\varepsilon}_{px}$. Substituting $\sigma_b = \sigma_b(\varepsilon_{px})$ according to (20.66), we obtain an explicit formula for the stress σ_x as a function of the plastic strain ε_{px}. The total strain is of course obtained by adding the elastic strain $\sigma_x(\varepsilon_{px})/E$ to the plastic strain ε_{px}.

For example, during monotonic tensile loading ($\operatorname{sgn}\dot{\varepsilon}_{px} = 1$) that starts from the initial virgin state ($\varepsilon_{p0} = 0$, $\sigma_{b0} = 0$), we have

$$\sigma_x(\varepsilon_{px}) = \sigma_0 + \sigma_b(\varepsilon_{px}) = \sigma_0 + \frac{H_K}{\gamma}\left[1 - \exp(-\gamma\varepsilon_{px})\right] \tag{20.67}$$

$$\varepsilon_x(\varepsilon_{px}) = \frac{1}{E}\sigma_x(\varepsilon_{px}) + \varepsilon_{px} \tag{20.68}$$

This equations provide a parametric description of the stress-strain diagram plotted by the solid curve in Figure 20.3(a). For $\varepsilon_{px} \to \infty$, the stress asymptotically approaches a limit value $\sigma_0 + H_K/\gamma$. This means that the hardening process becomes saturated and the back stress cannot exceed $\sigma_{b\infty} = H_K/\gamma = 50$ MPa. This observation helps to clarify the physical meaning of the parameter γ.

If the loading direction is reversed at plastic strain ε_{p1} and back stress $\sigma_{b1} = \sigma_{b\infty}[1 - \exp(-\gamma\varepsilon_{p1})]$, the yielding in compression starts at $\sigma_x = \sigma_{b1} - \sigma_0$ and the subsequent evolution of stress is given by

$$\sigma_x(\varepsilon_{px}) = -\sigma_0 - \sigma_{b\infty} + (\sigma_{b1} + \sigma_{b\infty})\exp\left[\gamma(\varepsilon_{px} - \varepsilon_{p1})\right] \tag{20.69}$$

For $\varepsilon_{px} \to -\infty$, the stress approaches the asymptotic limit $-\sigma_0 - \sigma_{b\infty}$, which has the same magnitude as the asymptotic limit that corresponds to monotonic loading

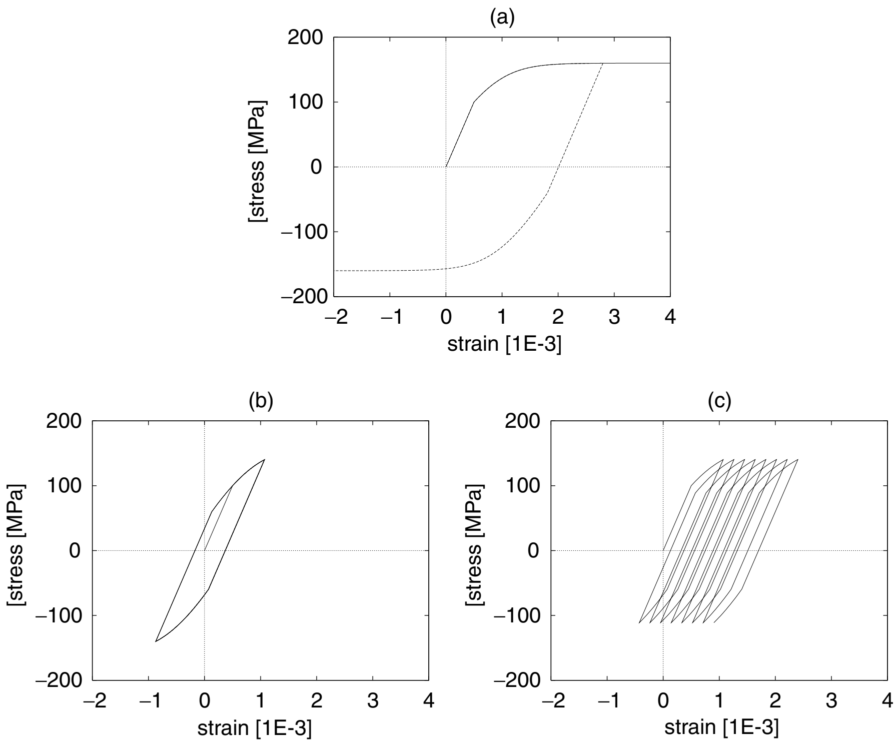

Figure 20.3 Stress-strain diagrams for a model with Armstrong–Frederick kinematic hardening: (a) monotonic loading and load reversal, (b) cyclic loading with a zero mean stress, (c) cyclic loading with a positive mean stress

from the virgin state; see the dashed curve in Figure 20.3(a). So, even though the hardening is kinematic and the onset of yielding upon load reversal occurs at a stress smaller in magnitude than the largest stress previously reached, the memory effect, after a sufficiently long monotonic yielding, fades away and the stress approaches a value independent of the previous loading history. This characteristic feature of the Armstrong–Frederick model can be used to check the suitability of that model for a particular material. In the general triaxial case, the state of saturated hardening (which is never reached exactly, only approached asymptotically) is characterized by a vanishing rate of the back stress. Substituting $\dot{\boldsymbol{\sigma}}_b = \mathbf{0}$ into (20.64), we obtain

$$\boldsymbol{\sigma}_b = \sqrt{\frac{2}{3}} \frac{H_K}{\gamma} \frac{\dot{\boldsymbol{\varepsilon}}_p}{\|\dot{\boldsymbol{\varepsilon}}_p\|} \tag{20.70}$$

from which

$$\|\boldsymbol{\sigma}_b\| = \sqrt{\frac{2}{3}} \frac{H_K}{\gamma} = \sqrt{\frac{2}{3}} \sigma_{b\infty} \tag{20.71}$$

So the parameter $\sigma_{b\infty}$ scaled by $\sqrt{2/3}$ characterizes the maximum distance by which the center of the yield surface can move in the stress space.

The stress-strain diagrams corresponding to cyclic loading, with stress varying in the range from −140 MPa to 140 MPa and from −110 MPa to 140 MPa are shown in Figures 20.3(b),(c). For symmetric stress cycles, the response immediately stabilizes and the strain history becomes periodic (Figure 20.3(b)). However, for stress cycles oscillating around a nonzero mean value, the net increment of plastic strain over one cycle is not zero (Figure 20.3(c)). This phenomenon, indeed observed in experiments, is called *ratchetting*. Note that this type of behavior could not be captured by the linear Melan–Prager hardening rule because a stabilized response ensues from this rule even if the mean value of stress is nonzero. In reality, of course, the net plastic strain increment over one cycle does not remain constant, and further refinements of the basic Armstrong–Frederick rule are needed to take that into account. □

20.1.5 General Hardening

All the hardening models discussed so far can be covered by the general yield condition

$$f(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0 \tag{20.72}$$

and the general hardening law

$$\dot{\boldsymbol{\kappa}} = \dot{\lambda} \boldsymbol{k}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{20.73}$$

where $\boldsymbol{\kappa}$ collects the hardening variables and $\boldsymbol{k}$ is a suitable function of the hardening variables and the stress. This function is in (20.73) multiplied by the plastic multiplier rate $\dot{\lambda}$, to make sure that the response of the model is not affected by any rescaling of the 'time' parameter, i.e. that the model remains rate-independent.

The mixed hardening model from Section 20.1.3 requires a scalar variable controling the isotropic hardening and a second-order tensor controling the kinematic hardening. The most straightforward choice would be $\boldsymbol{\kappa} = (\boldsymbol{\sigma}_b, \kappa)$, where $\boldsymbol{\sigma}_b$ is the back stress and κ is the cumulative plastic strain. However, the thermodynamic formulation of the model (Chapter 23) is more transparent if all the hardening variables have the dimension of strain. Therefore, we introduce a tensorial variable $\boldsymbol{\varepsilon}_b = \boldsymbol{\sigma}_b/\bar{H}_K$, and we set $\boldsymbol{\kappa} = (\boldsymbol{\varepsilon}_b, \kappa)$. The yield condition can be written in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_b, \kappa) \equiv F(\boldsymbol{\sigma} - \bar{H}_K \boldsymbol{\varepsilon}_b) - \bar{h}(\kappa) = 0 \tag{20.74}$$

The evolution of the hardening variables is defined by the rate equations

$$\dot{\boldsymbol{\varepsilon}}_b = \dot{\lambda} \boldsymbol{f}_\sigma(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_b) \tag{20.75}$$

$$\dot{\kappa} = \dot{\lambda} k(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_b, \kappa) \tag{20.76}$$

with the initial conditions $\boldsymbol{\varepsilon}_b = \mathbf{0}$ and $\kappa = 0$. With $k(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_b, \kappa)$ defined by (20.19) or (20.20), equation (20.76) covers the strain hardening and the work hardening hypotheses. Equation (20.75) has exactly the same form as the associated flow rule (20.25), and so the rate $\dot{\boldsymbol{\varepsilon}}_b$ is equal to the plastic strain rate $\dot{\boldsymbol{\varepsilon}}_p$.

Even though the values of $\boldsymbol{\varepsilon}_p$ and $\boldsymbol{\varepsilon}_b$ always remain equal, it is useful to consider them as two different physical quantities. This facilitates the generalization to nonlinear kinematic hardening. For example, the Armstrong–Frederick hardening rule (20.64) is obtained if (20.75) is replaced by

$$\dot{\boldsymbol{\varepsilon}}_b = \dot{\lambda} \left(\boldsymbol{f}_\sigma - \gamma \sqrt{\frac{2}{3}} \, \|\boldsymbol{f}_\sigma\| \, \boldsymbol{\varepsilon}_b \right) \tag{20.77}$$

To set up the consistency condition, we need to express the rate of the yield function in terms of the rates $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\kappa}}$. The notation is greatly simplified by the abstract operator $\bullet$, denoting the scalar product of $\dot{\boldsymbol{\kappa}}$ with the partial gradient $\boldsymbol{f}_\kappa = \partial f / \partial \boldsymbol{\kappa}$ (or any other similarly structured object). For instance, if $\boldsymbol{\kappa} = (\boldsymbol{\varepsilon}_b, \kappa)$, this operation must be interpreted as

$$\frac{\partial f}{\partial \boldsymbol{\kappa}} \bullet \dot{\boldsymbol{\kappa}} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}_b} : \dot{\boldsymbol{\varepsilon}}_b + \frac{\partial f}{\partial \kappa} \dot{\kappa} \tag{20.78}$$

Using this compact notation, we can write the consistency condition in the convenient form

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\kappa}} \bullet \dot{\boldsymbol{\kappa}} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} + \boldsymbol{f}_\kappa \bullet \dot{\boldsymbol{\kappa}} = 0 \tag{20.79}$$

After the usual substitutions from (20.24), (20.25) and (20.73), we get the formulae for the plastic multiplier rate,

$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma - \boldsymbol{f}_\kappa \bullet \boldsymbol{k}} \tag{20.80}$$

and for the elastoplastic stiffness tensor,

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{\boldsymbol{D}_e : \boldsymbol{f}_\sigma \otimes \boldsymbol{f}_\sigma : \boldsymbol{D}_e}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma - \boldsymbol{f}_\kappa \bullet \boldsymbol{k}} \tag{20.81}$$

Local uniqueness is guaranteed if $\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma - \boldsymbol{f}_\kappa \bullet \boldsymbol{k} > 0$.

20.2 DRUCKER'S POSTULATE AND UNIQUENESS

In previous chapters, we have repeatedly encountered the postulate of maximum plastic dissipation, which is equivalent to the conditions of convexity and normality and plays a key role in the derivation of powerful bound theorems of plastic analysis. Recall that this postulate is not a universally valid physical principle; it characterizes a certain class of material models with convenient mathematical properties. These so-called *standard materials* include many important models used in the engineering practice.

Another special class of elastoplastic models was defined by Drucker (1952, 1959) and called by him *stable materials*. Stability in Drucker's sense is stronger than convexity and normality. It permits us to extend the global uniqueness theorem from Section 16.3 to materials for which the yield surface evolves during the plastic flow, and showing that the response of a structure is unique not only in terms of stress but also in terms of strain and displacement.

Drucker's definition of stable materials can be motivated by the analysis of the work done in a quasi-cyclic process. Consider an infinitesimal volume of an elastoplastic material in a state characterized by stress $\boldsymbol{\sigma}^*$ and strain $\boldsymbol{\varepsilon}^*$. Suppose that, during the time interval $[0, T]$, an 'external agency' slowly applies some additional stress and then removes it. If the material remains elastic, it returns to its initial state. However, when plastic yielding takes place, the final strain $\boldsymbol{\varepsilon}(T)$ in general differs from the initial strain $\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}^*$, even if the additional stress is completely removed, i.e. if $\boldsymbol{\sigma}(T) = \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}^*$. This is why such a process is called a *stress quasi-cycle*.

Drucker's stability postulate requires the *excessive work* (or net work)

$$\Delta W = \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}(t)\,\mathrm{d}t \tag{20.82}$$

done by the additional stress, to be non-negative in any stress quasi-cycle. The physical idea behind this requirement is that if ΔW is negative, the 'external agency' can gain energy from the material and use it for further amplification of the disturbances. The postulate also tacitly assumes that the stress $\boldsymbol{\sigma}^*$ remains plastically admissible, at least if the stress $\boldsymbol{\sigma}(t)$ stays in some neighborhood of $\boldsymbol{\sigma}^*$. This assumption guarantees that the quasi-cycle can be closed in the stress space.

The strain increments can be decomposed into the elastic and plastic parts, and this decomposition is transfered to the excessive work,

$$\begin{aligned}\Delta W &= \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}(t)\,\mathrm{d}t \qquad (20.83)\\ &= \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}_e(t)\,\mathrm{d}t + \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}_p(t)\,\mathrm{d}t = \Delta W_e + \Delta W_p\end{aligned}$$

Due to the reversibility of elastic processes, the elastic strain $\boldsymbol{\varepsilon}_e(t) = \boldsymbol{C}_e : \boldsymbol{\sigma}(t)$ returns at the end of the cycle to its initial value, and the elastic part of the excessive work ΔW_e vanishes. This can be formally proven as follows:

$$\begin{aligned}\Delta W_e &= \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}_e(t)\,\mathrm{d}t = \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \boldsymbol{C}_e : \dot{\boldsymbol{\sigma}}(t)\,\mathrm{d}t \qquad (20.84)\\ &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\boldsymbol{\sigma}(t) : \boldsymbol{C}_e : \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^* : \boldsymbol{C}_e : \boldsymbol{\sigma}(t)\right]\mathrm{d}t\\ &= \tfrac{1}{2}\boldsymbol{\sigma}(T) : \boldsymbol{C}_e : \boldsymbol{\sigma}(T) - \tfrac{1}{2}\boldsymbol{\sigma}(0) : \boldsymbol{C}_e : \boldsymbol{\sigma}(0) - \boldsymbol{\sigma}^* : \boldsymbol{C}_e : [\boldsymbol{\sigma}(T) - \boldsymbol{\sigma}(0)] = 0\end{aligned}$$

Consequently, the excessive work can be computed as the work done by the additional stress on the plastic strain increments;

$$\Delta W = \Delta W_p = \int_0^T [\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^*] : \dot{\boldsymbol{\varepsilon}}_p(t)\,\mathrm{d}t \tag{20.85}$$

In the absence of plastic flow, ΔW_p always vanishes, and the stress quasi-cycle is a true cycle, because both stress and strain return to their initial values. If ΔW_p is strictly positive in any stress quasi-cycle that is not purely elastic, the material is called *stable in Drucker's sense*. If ΔW_p is always non-negative, but vanishes for some stress quasi-cycle that involves plastic flow, the material is called *neutrally stable in Drucker's sense*. Both stable and neutrally stable materials are classified as *materials satisfying Drucker's postulate*.

The previous definitions are physically motivated but, for a given material model, it would be hard to check the sign of ΔW_p in all possible stress quasi-cycles. Fortunately, there are simpler criteria of stability in Drucker's sense. Stable materials are those that satisfy the inequalities

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\varepsilon}}_p \geq 0 \tag{20.86}$$

$$\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}_p > 0 \tag{20.87}$$

for any stress $\boldsymbol{\sigma}$ and stress rate $\dot{\boldsymbol{\sigma}}$ generating a nonzero plastic strain rate $\dot{\boldsymbol{\varepsilon}}_p$, and for any plastically admissible stress $\boldsymbol{\sigma}^*$.

Condition (20.86) is equivalent to the postulate of maximum plastic dissipation, which is in turn equivalent to the conditions of convexity and normality. Condition (20.87) makes sure that the elastic domain during plastic flow expands, at least locally (i.e. in a neighborhood of the current stress state), so that it is possible to return to the stress state from which the stress quasi-cycle started. Consequently, (20.87) can be considered as the definition of hardening (in the narrow sense, meaning the opposite of softening). In summary, material stability in Drucker's sense is equivalent to convexity, normality and hardening. Elastic-perfectly plastic materials satisfying convexity and normality are only neutrally stable.

Example 20.6: Show that, for elastic-perfectly plastic materials satisfying convexity and normality, (20.87) turns into an equality.

Solution: For elastic-perfectly plastic materials, the yield function depends only on the stress, and the rate of its change is

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} \tag{20.88}$$

Substituting the associated flow rule (15.34) into the left-hand side of (20.87) and using (20.88), we obtain

$$\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}_p = \dot{\boldsymbol{\sigma}} : \boldsymbol{f}_\sigma \dot{\lambda} = \dot{f} \dot{\lambda} \tag{20.89}$$

According to the consistency condition (15.36), the product $\dot{f}\dot{\lambda}$ vanishes, and so $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}_p = 0$. □

Example 20.7: Check whether an associated elastoplasticity model with isotropic hardening satisfies condition (20.87).

Solution: Using the rate form of the elastic stress-strain law (20.24) and the associated flow rule (20.25), we can write

$$\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\sigma}} = \dot{\lambda}\boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma) = \dot{\lambda}\left(\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma\right) \tag{20.90}$$

During plastic yielding we have $\dot{\lambda} > 0$, and so the sign of $\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}_p$ depends on the expression in the parentheses on the right-hand side of (20.90). Since $\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma$ is always positive, condition (20.87) is equivalent to

$$\dot{\lambda} < \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma} \tag{20.91}$$

We recognize that the expression on the right-hand side of (20.91) corresponds to the formula for $\dot{\lambda}$ valid in perfect elastoplasticity, which confirms the result of the previous example. For associated elastoplasticity with isotropic hardening, the rate of the plastic multiplier is given by (20.30), and (20.91) can be rewritten as

$$\frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma + \bar{H}k} < \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{f}_\sigma} \tag{20.92}$$

In Section 20.1 we have established condition (20.37), which is necessary for uniqueness of the model response on the local level of one material point. If it is violated, the stress rate is not uniquely determined by the strain rate and the model cannot be used. Therefore, we can restrict attention to the case of a positive denominator on the left-hand side of (20.92). The numerator is also positive, or else the assumption of plastic yielding would not be valid. Consequently, (20.92) is equivalent to

$$\bar{H}k > 0 \tag{20.93}$$

Since k is a positive scaling constant, the model satisfies condition (20.87) if and only if the plastic modulus $\bar{H}$ is positive. So, Drucker's definition of hardening is consistent with the 'natural' definition in terms of the plastic modulus. □

The result of the previous example can be easily extended to the general class of hardening models described in Section 20.1.5. The plastic multiplier rate $\dot{\lambda}$ is in the general case given by (20.80), and so (20.93) is replaced by

$$-\boldsymbol{f}_\kappa \bullet \boldsymbol{k} > 0 \tag{20.94}$$

Drucker's postulate has important implications for global uniqueness. For stable materials, the structural response remains unique not only in terms of stresses but also in terms of strains and displacements. To show that, let us return to the proof of the global uniqueness theorem for perfectly plastic materials given in Section 16.3. The key step was the proof of the lemma asserting that if $\Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^A - \dot{\boldsymbol{\sigma}}^B$ is the difference between two stress rates and $\Delta\dot{\boldsymbol{\varepsilon}}_p = \dot{\boldsymbol{\varepsilon}}_p^A - \dot{\boldsymbol{\varepsilon}}_p^B$ is the difference between the corresponding plastic strain rates, then $\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}_p \geq 0$. Recall that both solutions labeled by A and B start from the same initial stress state. The derivation of the lemma was based on the identity

$$\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}_p = \Delta\dot{\boldsymbol{\sigma}} : \boldsymbol{f}_\sigma\,\Delta\dot{\lambda} = \Delta\dot{f}\,\Delta\dot{\lambda} \tag{20.95}$$

For hardening materials, $\Delta\dot{\boldsymbol{\sigma}} : \boldsymbol{f}_\sigma$ is not equal to $\Delta\dot{f}$, because the rate of the yield function involves an additional term due to hardening:

$$\dot{f} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} + \boldsymbol{f}_\kappa \bullet \dot{\boldsymbol{\kappa}} = \boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} + \boldsymbol{f}_\kappa \bullet \boldsymbol{k}\dot{\lambda} \tag{20.96}$$

Let us denote $\tilde{H} = -\boldsymbol{f}_\kappa \bullet \boldsymbol{k}$ and rewrite (20.96) as

$$\boldsymbol{f}_\sigma : \dot{\boldsymbol{\sigma}} = \dot{f} + \tilde{H}\dot{\lambda} \tag{20.97}$$

Since $\boldsymbol{f}_\sigma$ and $\tilde{H}$ are the same for both solutions A and B, the difference between (20.97) written for $\boldsymbol{\sigma}^A$, $\dot{f}^A$, $\dot{\lambda}^A$ and is $\boldsymbol{\sigma}^B$, $\dot{f}^B$, $\dot{\lambda}^B$

$$\boldsymbol{f}_\sigma : \Delta\dot{\boldsymbol{\sigma}} = \Delta\dot{f} + \tilde{H}\Delta\dot{\lambda} \tag{20.98}$$

Consequently, we can write

$$\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}_p = \Delta\dot{\boldsymbol{\sigma}} : \boldsymbol{f}_\sigma \, \Delta\dot{\lambda} = \Delta\dot{f}\,\Delta\dot{\lambda} + \tilde{H}(\Delta\dot{\lambda})^2 \tag{20.99}$$

The product $\Delta\dot{f}\,\Delta\dot{\lambda}$ was in Section 16.3 shown to be non-negative. If the material is stable in Drucker's sense, it follows from (20.94) that $\tilde{H} = -\boldsymbol{f}_\kappa \bullet \boldsymbol{k}$ is indeed a positive variable; it can be considered as a generalized hardening modulus. Therefore, the expression $\tilde{H}(\Delta\dot{\lambda})^2$ is positive, unless $\Delta\dot{\lambda} = 0$, in which case $\Delta\dot{\boldsymbol{\varepsilon}}_p = \boldsymbol{f}_\sigma \, \Delta\dot{\lambda} = \boldsymbol{0}$. Consequently, the product $\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}_p$ is always non-negative, and vanishes only if $\Delta\dot{\boldsymbol{\varepsilon}}_p = \boldsymbol{0}$. As already mentioned in Section 16.3, the expression $\Delta\dot{\boldsymbol{\sigma}} : \boldsymbol{C}_e : \Delta\dot{\boldsymbol{\sigma}}$ is always nonnegative and vanishes only if $\Delta\dot{\boldsymbol{\sigma}} = \boldsymbol{0}$. Combining these results, we conclude that

$$\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = \Delta\dot{\boldsymbol{\sigma}} : \boldsymbol{C}_e : \Delta\dot{\boldsymbol{\sigma}} + \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}}_p \tag{20.100}$$

is always non-negative and vanishes only if $\Delta\dot{\boldsymbol{\sigma}} = \boldsymbol{0}$ and $\Delta\dot{\boldsymbol{\varepsilon}}_p = \boldsymbol{0}$. Since $\Delta\dot{\boldsymbol{\sigma}}$ is self-equilibrated and $\Delta\dot{\boldsymbol{\varepsilon}}$ is compatible, we have

$$\int_V \Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} \,\mathrm{d}V = 0 \tag{20.101}$$

and, due to the non-negativity of the integrand, this is possible only if $\Delta\dot{\boldsymbol{\sigma}} : \Delta\dot{\boldsymbol{\varepsilon}} = 0$. For a perfectly plastic material we could conclude only that $\dot{\boldsymbol{\sigma}}^A = \dot{\boldsymbol{\sigma}}^B$, but for a hardening material we have also $\dot{\boldsymbol{\varepsilon}}_p^A = \dot{\boldsymbol{\varepsilon}}_p^B$, and from $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{C}_e : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}_p$ it follows that $\dot{\boldsymbol{\varepsilon}}^A = \dot{\boldsymbol{\varepsilon}}^B$. The response remains unique in terms of the stress and strain histories. Uniqueness of the displacement history follows from the uniqueness of the strain history, provided that the kinematic boundary conditions are sufficient to suppress rigid-body motions.

20.3 NONASSOCIATED FLOW

From the purely mathematical point of view it would be convenient to use only associated flow rules. Unfortunately, such rules do not always describe the real physical processes in a sufficiently realistic manner, especially if the material is pressure-sensitive. For example, a flow rule associated with the Mohr–Coulomb yield condition for soils usually overestimates the volumetric part of plastic strain. The source of the problem can be illustrated by a simple mechanical model of dry Coulomb friction. Slip on a contact surface between two bodies in Figure 20.4(a) is initiated when

$$|T| = -\mu N \tag{20.102}$$

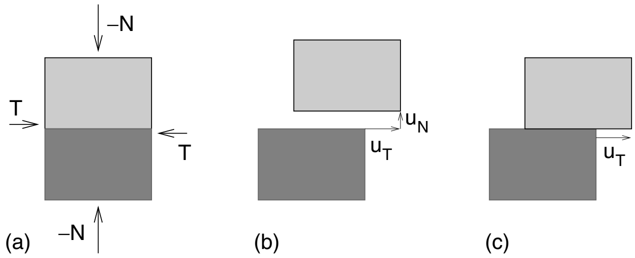

Figure 20.4 Coulomb friction model

where T is the tangential component of the contact force, N is the normal contact force ($N < 0$), and μ is the static coefficient of friction. The corresponding yield function can be written as

$$f(T, N) = |T| + \mu N \tag{20.103}$$

Kinematic variables work-conjugate to forces T and N are the opening (or gap) u_N and the relative slip u_T (the normal and tangential component of the relative displacement between the bodies), shown in Figure 20.4(b). The associated flow rule would read

$$\dot{u}_T = \dot{\lambda}\frac{\partial f}{\partial T} = \dot{\lambda}\ \text{sign}\ T \tag{20.104}$$

$$\dot{u}_N = \dot{\lambda}\frac{\partial f}{\partial N} = \dot{\lambda}\mu \tag{20.105}$$

This means that an associated flow should take place with a fixed ratio $\dot{u}_N/|\dot{u}_T| = \mu$, i.e. the normal relative displacement would be proportional to the tangential relative displacement. However, this is at odds with the 'natural' assumption that the bodies remain in contact and only tangential slip takes place as shown in Figure 20.4(c). A model exhibiting such behavior requires a nonassociated flow rule derived from a plastic potential

$$g(T, N) = |T| \tag{20.106}$$

that differs from the yield function (20.103).

The friction model with a zero gap is the most rudimentary one, based on our macroscopic 'everyday experience'. It assumes that the surfaces are 'rough' but the asperities are so small compared to the scale of observation that we still consider the surfaces as ideal planes that allow sliding in the tangential direction without any normal displacement. The roughness is reflected only by the frictional resistance that develops under normal pressure. A closer look at the interface motivates a different type of model that reflects the asperities directly as deviations from the ideal shape of the surface. One strongly idealized model of this type is sketched in Figure 20.5(a). The interface is represented by a zig-zag line consisting of straight segments with a constant deviation from the 'macroscopic' contact line. Locally, on the level of asperities, we could assume that the contact is frictionless and only forces normal to the contact segments can be transmitted. Due to the inclination of the segments, the interface can even transmit (without any slip) a force that is not perpendicular to the mean contact line, provided that its tangential component is sufficiently small; see Figure 20.5(b). Slip is initiated when the deviation of the global contact force from the normal to the mean contact line equals the deviation of the contact segments from

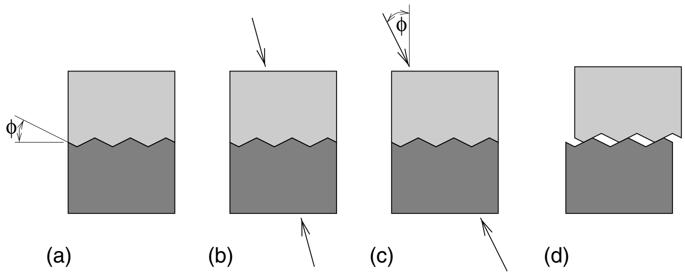

Figure 20.5 Friction model with asperities

the mean contact line (Figure 20.5(c)), i.e. if

$$\frac{|T|}{-N} = \tan\phi \tag{20.107}$$

This is the same condition as (20.102) if we identify $\tan\phi$ with the friction coefficient μ. The angle ϕ can be called the *friction angle.* A remarkable feature of the present model is that the slip not only has a component parallel to the mean contact line but also a component normal to it, and their ratio is obviously $\tan\phi = \mu$; see Figure 20.5(d). The model therefore obeys the associated flow rule (20.104)–(20.105).

The two models just described are certainly very schematic but they offer some insight into the behavior of frictional materials. Note that the model with geometric asperities would only give an associated flow in a certain range limited by the size of the asperities. After a sufficiently large relative displacement the behavior would approach that of the model with only tangential slip. The situation is also complicated by the fact that real asperities have variable sizes and shapes, and that they experience some plastic deformation and damage during sliding.

A number of additional factors should be taken into account when extending the models of friction on an interface between two blocks to materials that have no pre-existing internal interfaces. Here, a part of a potential slip plane may already be slipping and resisting the slip by friction, and the rest of that plane still retains cohesion. The cohesion is reflected by an additional constant term in the yield function. If the plastic deformation were exclusively due to slip, the plastic strain would be purely deviatoric. However, the presence of a relative displacement component normal to the sliding plane results into plastic changes of volume. The simple mechanical model of a surface with rigid asperities predicts an increase of volume during plastic flow. This effect is called *dilatancy* (occasionally also dilatation or dilation) and is observed for example in concrete under low confinement, in overconsolidated clays, or in dense sands. On the other hand, some materials such as concrete under very high confinement, normally consolidated clays, or loose sands exhibit a plastic decrease of volume, which is called *contractancy* (or negative dilatancy). It is often possible to assume that the flow is associated in the projection onto the deviatoric plane and only its volumetric part is nonassociated. A suitable plastic potential can be derived from the yield function by modifying only the term that reflects pressure sensitivity of the material. For example, from the Drucker–Prager criterion (15.20) we could derive a plastic potential

$$g(I_1, J_2) = \alpha_\psi I_1 + \sqrt{J_2} \tag{20.108}$$

where α_ψ is the *dilatancy coefficient*, in general different from the friction coefficient α that appears in the yield function. For $\alpha_\psi = \alpha$ we have an associated flow, for $\alpha_\psi = 0$ a purely deviatoric flow, for $\alpha_\psi \in (0, \alpha)$ a nonassociated dilatant flow, and for $\alpha_\psi < 0$ a contractive flow. In (20.108), the constant term $-\tau_0$ from the original yield function has been omitted because it has no effect on the gradient $\partial g/\partial\boldsymbol{\sigma}$ that determines the direction of plastic flow.

It is not difficult to generalize the formulae for the rate of plastic multiplier and the elastoplastic stiffness to the case of a nonassociated flow described by the flow rule

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{20.109}$$

To simplify the notation, let us denote the gradient of the plastic potential as $\boldsymbol{g}_\sigma$. Substituting (20.24), (20.73) and (20.109) into the consistency condition we get

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \kappa} \bullet \dot{\kappa} = \boldsymbol{f}_\sigma : \boldsymbol{D}_e : (\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{g}_\sigma) + \boldsymbol{f}_\kappa \bullet k\dot{\lambda} = 0 \tag{20.110}$$

from which

$$\dot{\lambda} = \frac{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{g}_\sigma - \boldsymbol{f}_\kappa \bullet k} \tag{20.111}$$

The elastoplastic stiffness tensor is given by

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{\boldsymbol{D}_e : \boldsymbol{g}_\sigma \otimes \boldsymbol{f}_\sigma : \boldsymbol{D}_e}{\boldsymbol{f}_\sigma : \boldsymbol{D}_e : \boldsymbol{g}_\sigma - \boldsymbol{f}_\kappa \bullet k} \tag{20.112}$$

Example 20.8: Derive the specific expression for the elastoplastic stiffness of a Drucker–Prager material with isotropic strain hardening and a nonassociated flow rule derived from the plastic potential (20.108). Discuss local uniqueness.

Solution: Using the chain rule from Appendix D.3, we obtain the gradient of the Drucker–Prager yield function,

$$\boldsymbol{f}_\sigma = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial J_2}\frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \alpha\boldsymbol{\delta} + \frac{1}{2\sqrt{J_2}}\boldsymbol{s} \tag{20.113}$$

The gradient of the plastic potential,

$$\boldsymbol{g}_\sigma = \alpha_\psi\boldsymbol{\delta} + \frac{1}{2\sqrt{J_2}}\boldsymbol{s} \tag{20.114}$$

is obtained simply by replacing the friction coefficient α by the dilatancy coefficient α_ψ. For isotropic strain hardening we have $\boldsymbol{f}_\kappa \bullet k = -\bar{H}k$, where

$$k = \sqrt{\frac{2}{3}}\|\boldsymbol{g}_\sigma\| = \sqrt{\frac{2}{3}\left(\alpha_\psi\boldsymbol{\delta} + \frac{1}{2\sqrt{J_2}}\boldsymbol{s}\right) : \left(\alpha_\psi\boldsymbol{\delta} + \frac{1}{2\sqrt{J_2}}\boldsymbol{s}\right)} = \sqrt{2\alpha_\psi^2 + \frac{1}{3}} \tag{20.115}$$

because $\boldsymbol{\delta} : \boldsymbol{\delta} = 3$, $\boldsymbol{\delta} : \boldsymbol{s} = 0$, and $\boldsymbol{s} : \boldsymbol{s} = 2J_2$. Substituting into the general formula (20.112) and using the relations $\boldsymbol{D}_e : \boldsymbol{\delta} = 3K\boldsymbol{\delta}$ and $\boldsymbol{D}_e : \boldsymbol{s} = 2G\boldsymbol{s}$, we obtain the desired expression

$$\boldsymbol{D}_{ep} = \boldsymbol{D}_e - \frac{9K^2\alpha\alpha_\psi\boldsymbol{\delta}\otimes\boldsymbol{\delta} + \dfrac{3KG}{\sqrt{J_2}}(\alpha\boldsymbol{s}\otimes\boldsymbol{\delta} + \alpha_\psi\boldsymbol{\delta}\otimes\boldsymbol{s}) + \dfrac{G}{J_2}\boldsymbol{s}\otimes\boldsymbol{s}}{9K\alpha\alpha_\psi + G + \bar{H}\sqrt{2\alpha_\psi^2 + \dfrac{1}{3}}} \tag{20.116}$$

An inspection of the denominator reveals that the condition of local uniqueness $9K\alpha\alpha_\psi + G + \bar{H}k > 0$ can be violated even for a non-softening material ($\bar{H} \geq 0$) if the flow is contractive ($\alpha_\psi < 0$). The critical value of α_ψ for a perfectly plastic material is

$$\alpha_{\psi,crit} = -\frac{G}{9K\alpha} = -\frac{1}{6\alpha}\frac{1-2\nu}{1+\nu} \tag{20.117}$$

□

Although for plasticity models with an associated flow rule the tangential stiffness tensor is symmetric, nonsymmetry should characterize a more realistic model. For two-dimensional behavior, this can be illustrated by a model of frictionally sliding blocks connected by springs, presented in Bažant and Cedolin (1991, Section 10.7, Figure 10.22). The nonsymmetry reflecting friction, however, must be of a form that causes no material instability. Generalizing from Mandel's (1964a) ingeneous example of a spring-loaded frictionally-sliding block, Bažant (1980) formulated the sufficient (albeit not necessary) conditions under which the tangential stiffness tensor can be nonsymmetric while causing no material instability.

Comparison with the aforementioned spring-block model further reveals that the modeling of internal friction as an influence of I_1 on $\sqrt{J_2}$, as done by the Drucker–Prager criterion, is a gross simplification. It cannot reflect the fact that frictional slip occurs on one or several preferred planes. It is an advantage of the microplane model (to be presented in Section 25.2) that it can capture this fact.

20.4 NON-SMOOTH AND MULTI-SURFACE PLASTICITY

Another issue that deserves attention is the treatment of singularities of a yield surface, e.g. of the corners (edges) of the Tresca hexagon (hexahedral prism). At such points, the yield surface does not have a uniquely defined normal because the gradient $\boldsymbol{f}_\sigma$ vanishes or even does not exist. If the flow rule is associated, the direction of plastic flow is not properly defined. These problems can be handled if the elastic domain bounded by a non-smooth yield surface is represented as an intersection of several domains bounded by smooth auxiliary surfaces.

20.4.1 Perfect Plasticity

We start by looking at the simple case of a perfectly elastoplastic material. For illustration, consider the criterion due to Tresca, with the yield function given by the product-like expression (15.14). The inequality $f(\boldsymbol{\sigma}) \leq 0$ is not an unambiguous description of the Tresca hexahedral prism because it also holds at points for which, for example, $\sigma_1 - \sigma_2 > 2\tau_0$, $\sigma_2 - \sigma_3 > 2\tau_0$, and $|\sigma_3 - \sigma_1| < 2\tau_0$. Therefore it is preferable not to describe the elastic domain in Figure 20.6 by a single inequality $f(\boldsymbol{\sigma}) < 0$ but by a set of three inequalities

$$f_1(\boldsymbol{\sigma}) \equiv (\sigma_2 - \sigma_3)^2 - 4\tau_0^2 < 0 \tag{20.118}$$

$$f_2(\boldsymbol{\sigma}) \equiv (\sigma_3 - \sigma_1)^2 - 4\tau_0^2 < 0 \tag{20.119}$$

$$f_3(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_2)^2 - 4\tau_0^2 < 0 \tag{20.120}$$

which checks the signs of functions f_1, f_2 and f_3 separately. Plastic flow occurs if at least one of these functions vanishes. The auxiliary yield surfaces for which

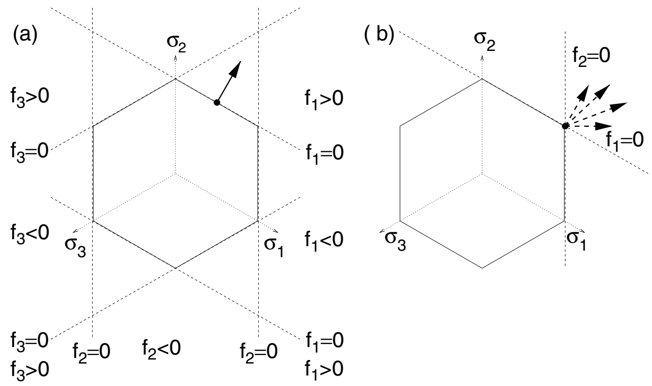

Figure 20.6 Deviatoric section of the Tresca yield surface: a) plastic flow at a regular point, b) plastic flow at a vertex

$f_I = 0$ are then called *active*. If, for example, only surface 1 is active ($f_1 = 0$ while $f_2 < 0$ and $f_3 < 0$, see Figure 20.6(a)), the stress state corresponds to a regular point of the yield surface, which has a uniquely defined normal determined by the gradient $\boldsymbol{f}_\sigma \equiv \partial f/\partial\boldsymbol{\sigma} = f_3 f_2 \partial f_1/\partial\boldsymbol{\sigma} + f_1\partial(f_2 f_3)/\partial\boldsymbol{\sigma} = f_3 f_2 \boldsymbol{f}_{1,\sigma}$, colinear with $\boldsymbol{f}_{1,\sigma} \equiv \partial f_1/\partial\boldsymbol{\sigma}$. However, it can happen that two surfaces are active simultaneously, e.g. $f_1 = 0$ and $f_2 = 0$ while $f_3 < 0$ (Figure 20.6(b)). Then the stress point is located at an edge of the hexahedral prism, where the original yield function f has a zero gradient, but the auxiliary functions f_1 and f_2 have nonzero gradients $\boldsymbol{f}_{1,\sigma}$ and $\boldsymbol{f}_{2,\sigma}$. It has already been explained in Section 15.2.1 that the postulate of maximum plastic dissipation remains valid if the direction of plastic flow is anywhere within the fan of directions between $\boldsymbol{f}_{1,\sigma}$ and $\boldsymbol{f}_{2,\sigma}$. This fan is formed by vectors that are linear combinations of $\boldsymbol{f}_{1,\sigma}$ and $\boldsymbol{f}_{2,\sigma}$ with non-negative coefficients. The corresponding flow rule reads

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}_1 \boldsymbol{f}_{1,\sigma} + \dot{\lambda}_2 \boldsymbol{f}_{2,\sigma} \tag{20.121}$$

where $\dot{\lambda}_1 \geq 0$ and $\dot{\lambda}_2 \geq 0$ are the rates of plastic multipliers associated with the auxiliary surfaces $f_1 = 0$ and $f_2 = 0$. To cover the general situation of yielding at an arbitrary point of the yield surface, we may write the flow rule as

$$\dot{\boldsymbol{\varepsilon}}_p = \sum_{I=1}^{N} \dot{\lambda}_I \boldsymbol{f}_{I,\sigma} \tag{20.122}$$

where N is the number of auxiliary yield surfaces. This is the famous *Koiter's rule* (Koiter, 1953b). A multiplier $\dot{\lambda}_I$ can be nonzero only if the stress point lies on the corresponding surface $f_I = 0$. This leads us to the generalized loading-unloading conditions

$$f_I \leq 0, \qquad \dot{\lambda}_I \geq 0, \qquad f_I \dot{\lambda}_I = 0, \qquad I = 1, 2, \ldots N \tag{20.123}$$

which again correspond to the Karush–Kuhn–Tucker conditions known from optimization theory (see Section 15.2.4). In $f_I \dot{\lambda}_I$ and similar expressions, the summation convention does not apply because the subscript I does not refer to tensorial components, but simply labels individual yield surfaces.

If, during a certain time interval, the Ith plastic multiplier is increasing ($\dot{\lambda}_I > 0$), the stress point must stay on the Ith auxiliary yield surface, and consequently $\dot{f}_I = 0$.

We thus obtain N consistency conditions

$$\dot{f}_I \dot{\lambda}_I = 0, \qquad I = 1, 2, \dots N \tag{20.124}$$

which can be exploited while computing the rates of plastic multipliers $\dot{\lambda}_I$, $I = 1, 2, \dots N$. Following the standard procedure, we can obtain the rates

$$\dot{f}_I = \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \left(\dot{\boldsymbol{\varepsilon}} - \sum_{J=1}^{N} \dot{\lambda}_J \boldsymbol{f}_{J,\sigma} \right), \qquad I = 1, 2, \dots N \tag{20.125}$$

Note that the rate $\dot{f}_I$ depends not only on $\dot{\lambda}_I$ but, in general, on all $\dot{\lambda}_J$, $J = 1, 2, \dots N$. If several yield surfaces are active simultaneously, the consistency conditions become coupled.

Suppose that the yield surfaces are renumbered such that surfaces number $1, 2, \dots M$ are currently active while $M+1, M+2, \dots N$ are not. The currently active surfaces are characterized by $f_I = 0$. They can either remain active, in which case $\dot{f}_I = 0$ and $\dot{\lambda}_I > 0$, or start unloading, in which case $\dot{f}_I < 0$ and $\dot{\lambda}_I = 0$. Both situations are covered by the consistency conditions

$$\dot{f}_I \le 0, \qquad \dot{\lambda}_I \ge 0, \qquad \dot{f}_I \dot{\lambda}_I = 0, \qquad I = 1, 2, \dots M \tag{20.126}$$

which are formally similar to (20.123). The difference is that (1) the values of the yield functions are replaced by their rates, and (2) conditions (20.123) hold for all the auxiliary yield surfaces while (20.126) only for the active ones.

It is convenient to introduce column matrices

$$\dot{\boldsymbol{f}} = \{\dot{f}_1, \dot{f}_2, \dots \dot{f}_M\}^T, \qquad \dot{\boldsymbol{\lambda}} = \{\dot{\lambda}_1, \dot{\lambda}_2, \dots \dot{\lambda}_M\}^T \tag{20.127}$$

and rewrite conditions (20.126) as

$$\dot{\boldsymbol{f}} \le \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}} \ge \boldsymbol{0}, \qquad \dot{\boldsymbol{f}}^T \dot{\boldsymbol{\lambda}} = 0 \tag{20.128}$$

Note that $\dot{\boldsymbol{f}}^T \dot{\boldsymbol{\lambda}} = \sum \dot{f}_I \dot{\lambda}_I$, and so the isolated condition $\dot{\boldsymbol{f}}^T \dot{\boldsymbol{\lambda}} = 0$ is not equivalent to $\dot{f}_I \dot{\lambda}_I$ vanishing for all $I = 1, 2, \dots M$ (no sum over M). However, the ensemble of conditions (20.128) is equivalent to (20.126) because, if all $\dot{f}_I$ are nonpositive and all $\dot{\lambda}_I$ are nonnegative, then each term $\dot{f}_I \dot{\lambda}_I$ is nonpositive and the sum of such terms can vanish only if each individual term vanishes.

According to (20.125), the rate $\dot{\boldsymbol{f}}$ can be expressed as

$$\dot{\boldsymbol{f}} = \boldsymbol{b} - \boldsymbol{A} \dot{\boldsymbol{\lambda}} \tag{20.129}$$

where $\boldsymbol{b}$ is a column matrix with entries $b_I = \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}}$, $I = 1, 2, \dots M$, and $\boldsymbol{A}$ is an $M \times M$ matrix with entries

$$a_{IJ} = \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{J,\sigma}, \qquad I, J = 1, 2 \dots M \tag{20.130}$$

Note that $a_{ij} = a_{ji}$, i.e. matrix $\boldsymbol{A}$ is symmetric. Substituting (20.129) into (20.128) we obtain the *linear complementarity problem* (LCP)

$$\boldsymbol{A}\dot{\boldsymbol{\lambda}} - \boldsymbol{b} \ge \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}} \ge \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}}^T (\boldsymbol{A}\dot{\boldsymbol{\lambda}} - \boldsymbol{b}) = \boldsymbol{0} \tag{20.131}$$

Linear complementarity problems have been extensively studied in optimization theory, because they are closely related to *quadratic programming* (minimization of quadratic functions subject to linear constraints). Sophisticated algorithms are available for the solution of such problems. The following theorem provides a necessary and sufficient condition of existence and uniqueness. Its proof can be found, for instance, in Cottle, Pang and Stone (1992).

Theorem 20.1
The LCP (20.131) has exactly one solution $\dot{\boldsymbol{\lambda}}$ for any vector $\boldsymbol{b}$ if and only if all the principal minors of matrix $\boldsymbol{A}$ are positive.

Recall that the principal minors of an $M \times M$ matrix are the determinants of all submatrices constructed by selecting a set $\mathcal{I} \subset \{1, 2, \dots M\}$ and deleting all the rows and columns whose number is contained in $\mathcal{I}$. For symmetric matrices, positivity of all the principal minors is equivalent to the *positive definiteness* of the matrix, i.e. to the condition that

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0 \qquad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{20.132}$$

For general matrices, positivity of all the principal minors is equivalent to the so called *P-positivity*, i.e. to the condition

$$\forall \boldsymbol{x} \neq \boldsymbol{0} \;\; \exists\, i \in \{1, 2, \dots M\} : \qquad x_i (\boldsymbol{A}\boldsymbol{x})_i > 0 \qquad \text{(no sum over } i\text{)} \tag{20.133}$$

Here, x_i is the ith component of vector $\boldsymbol{x}$, and $(\boldsymbol{A}\boldsymbol{x})_i$ is the ith component of vector $\boldsymbol{A}\boldsymbol{x}$. Condition (20.133) means that there is no nonzero vector $\boldsymbol{x}$ for which multiplication by $\boldsymbol{A}$ would reverse the signs of all its nonzero components.

Theorem 20.1 facilitates the checks of uniqueness for plasticity models with multiple yield surfaces. Let us now apply the theorem to our plasticity model with a composite yield surface. Matrix $\boldsymbol{A}$ given by (20.130) is symmetric, and it is positive definite if

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} &= \sum_{I,J=1}^{M} x_I a_{IJ} x_J = \sum_{I,J=1}^{M} x_I \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{J,\sigma} x_J \\ &= \left(\sum_{I=1}^{M} x_I \boldsymbol{f}_{I,\sigma} \right) : \boldsymbol{D}_e : \left(\sum_{J=1}^{M} x_J \boldsymbol{f}_{J,\sigma} \right) > 0 \qquad \forall \boldsymbol{x} \neq \boldsymbol{0} \end{aligned} \tag{20.134}$$

The final expression in (20.134) has the form $\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{f}$ where

$$\boldsymbol{f} = \sum_{I=1}^{M} x_I \boldsymbol{f}_{I,\sigma} \tag{20.135}$$

Since $\boldsymbol{D}_e$ is a positive definite tensor, we have $\boldsymbol{f} : \boldsymbol{D}_e : \boldsymbol{f} > 0$ whenever $\boldsymbol{f} \neq \boldsymbol{0}$. So the positive definiteness of $\boldsymbol{A}$ can be proven if the sum in (20.135) is nonzero for any set of coefficients x_I, $I = 1, 2, \dots M$, that are not simultaneously zero. This condition means that the gradients $\boldsymbol{f}_{I,\sigma}$, $I = 1, 2, \dots M$, must be linearly independent. We have derived the following theorem:

Theorem 20.2

Consider associated perfect plasticity with a composite yield surface described by a set of auxiliary surfaces. If, at any singular point, the normals to all those auxiliary yield functions that intersect at that point are linearly independent, then the model gives a unique response (in terms of stress and plastic strain evolution) to any prescribed strain evolution.

The condition of linear independence limits the number of yield surfaces that may intersect at the same point. In three spatial dimensions, the normals belong to the six-dimensional space of symmetric second-order tensors, and so the maximum admissible number of simultaneously active surfaces is six. For plane stress, this number reduces to three.

If at most one yield surface is active, the problem reduces to the standard one, and the solution is given by (15.40). Let us now explore the case of two initially active yield surfaces.

Example 20.9: Set up and solve the linear complementarity problem describing associated plastic flow at an edge of the Tresca hexahedral prism, assuming that the material is perfectly elastoplastic.

Solution: The elastic domain corresponding to the Tresca criterion is described by (20.118)–(20.120). At most, two surfaces can be active simultaneously. Without any loss of generality, let us assume that the stress point is at the edge characterized by

$$\sigma_2 - \sigma_3 = 2\tau_0 \tag{20.136}$$

$$\sigma_1 - \sigma_3 = 2\tau_0 \tag{20.137}$$

and so surfaces 1 and 2 are active; see Figure 20.6(b). Differentiating the yield function $f_1(\boldsymbol{\sigma})$ defined in (20.118), we obtain

$$\frac{\partial f_1}{\partial \boldsymbol{\sigma}} = \sum_{I=1}^{3} \frac{\partial f_1}{\partial \sigma_I} \frac{\partial \sigma_I}{\partial \boldsymbol{\sigma}} = 2(\sigma_2 - \sigma_3) \left(\frac{\partial \sigma_2}{\partial \boldsymbol{\sigma}} - \frac{\partial \sigma_3}{\partial \boldsymbol{\sigma}} \right) \tag{20.138}$$

According to formula (D.61) derived in Appendix D, we have

$$\frac{\partial \sigma_2}{\partial \boldsymbol{\sigma}} = \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 \qquad \text{and} \qquad \frac{\partial \sigma_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 \tag{20.139}$$

where $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are unit vectors in the principal directions associated with principal stresses σ_2 and σ_3, respectively. The gradient of f_1 is thus given by

$$\boldsymbol{f}_{1,\sigma} = 2(\sigma_2 - \sigma_3)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 - \boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \tag{20.140}$$

and a similar expression

$$\boldsymbol{f}_{2,\sigma} = 2(\sigma_1 - \sigma_3)(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1 - \boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \tag{20.141}$$

can be derived for the gradient of f_2.

Evaluating the entries of matrix $\boldsymbol{A}$, we have to deal with terms of the type $(\boldsymbol{n}_I \otimes \boldsymbol{n}_I) : \boldsymbol{D}_e : (\boldsymbol{n}_J \otimes \boldsymbol{n}_J)$. Such expressions are scalars, i.e. they are invariant with respect to the choice of the coordinate system, and they can be conveniently evaluated in the principal stress coordinates. For the isotropic elastic stiffness tensor $\boldsymbol{D}_e$ we obtain

$$(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) : \boldsymbol{D}_e : (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) = D^e_{2211} = K - \frac{2}{3}G \tag{20.142}$$

$$(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) : \boldsymbol{D}_e : (\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) = D^e_{3333} = K + \frac{4}{3}G \tag{20.143}$$

Now it is easy to evaluate

$$a_{11} = \boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} = 4(\sigma_2 - \sigma_3)^2(D^e_{2222} + D^e_{3333} - D^e_{2233} - D^e_{3322})$$
$$= 16G(\sigma_2 - \sigma_3)^2 \tag{20.144}$$
$$a_{12} = \boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} = 4(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3)(D^e_{2211} + D^e_{3333} - D^e_{2233} - D^e_{3311})$$
$$= 8G(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3) \tag{20.145}$$
$$a_{22} = \boldsymbol{f}_{2,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} = 4(\sigma_1 - \sigma_3)^2(D^e_{1111} + D^e_{3333} - D^e_{1133} - D^e_{3311})$$
$$= 16G(\sigma_1 - \sigma_3)^2 \tag{20.146}$$

Making use of (20.136) and (20.137), we finally obtain

$$\boldsymbol{A} = 32G\tau_0^2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{20.147}$$

This matrix is positive definite, and so the problem has a unique solution for any prescribed strain rate.

The first component of the column matrix $\boldsymbol{b}$ is given by

$$b_1 = \boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} = 2(\sigma_2 - \sigma_3)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 - \boldsymbol{n}_3 \otimes \boldsymbol{n}_3) : \boldsymbol{D}_e : (\dot{\varepsilon}_V \boldsymbol{\delta} + \dot{\boldsymbol{e}})$$
$$= 4\tau_0(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 - \boldsymbol{n}_3 \otimes \boldsymbol{n}_3) : (3K\dot{\varepsilon}_V \boldsymbol{\delta} + 2G\dot{\boldsymbol{e}}) \tag{20.148}$$

Realizing that $(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) : \boldsymbol{\delta} = \boldsymbol{n}_2 \cdot \boldsymbol{\delta} \cdot \boldsymbol{n}_2 = \boldsymbol{n}_2 \cdot \boldsymbol{n}_2 = 1$, $(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) : \dot{\boldsymbol{e}} = \boldsymbol{n}_2 \cdot \dot{\boldsymbol{e}} \cdot \boldsymbol{n}_2 = \dot{e}_{22}$, etc., we can simplify (20.148) to $b_1 = 8G\tau_0(\dot{e}_{22} - \dot{e}_{33})$. Here, $\dot{e}_{22}$ and $\dot{e}_{33}$ are the components of the deviatoric strain rate expressed in the principal stress coordinates. Evaluating b_2 in an analogous way, we finally obtain

$$\boldsymbol{b} = 8G\tau_0 \left\{ \begin{array}{c} \dot{e}_{22} - \dot{e}_{33} \\ \dot{e}_{11} - \dot{e}_{33} \end{array} \right\} \tag{20.149}$$

The linear complementarity problem (20.131) is now fully specified, and we can analyze its solutions. Three basic situations can be distinguished:

1. *Both surfaces remain active*, i.e. the stress point remains at the edge of the yield surface. In this case, $\dot{\lambda}_1$ and $\dot{\lambda}_2$ can be solved from the set of linear equations $\boldsymbol{A}\dot{\boldsymbol{\lambda}} = \boldsymbol{b}$ with $\boldsymbol{A}$ and $\boldsymbol{b}$ given by (20.147) and (20.149), respectively. The solution

$$\dot{\boldsymbol{\lambda}} = \frac{1}{12\tau_0} \left\{ \begin{array}{c} 2\dot{e}_{22} - \dot{e}_{11} - \dot{e}_{33} \\ 2\dot{e}_{11} - \dot{e}_{22} - \dot{e}_{33} \end{array} \right\} \tag{20.150}$$

   is admissible if both $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are non-negative, i.e. if

$$2\dot{e}_{22} - \dot{e}_{11} - \dot{e}_{33} \geq 0 \tag{20.151}$$

$$2\dot{e}_{11} - \dot{e}_{22} - \dot{e}_{33} \geq 0 \tag{20.152}$$

2. (a) Only *one surface remains active*, say surface 1. In this case, the stress point leaves the edge and moves to the regular part of the yield surface on which $f_1 = 0$. The rate $\dot{\lambda}_2$ vanishes, and $\dot{\lambda}_1$ is computed from $a_{11}\dot{\lambda}_1 = b_1$. The solution

$$\dot{\boldsymbol{\lambda}} = \frac{1}{8\tau_0}\begin{Bmatrix} \dot{e}_{22} - \dot{e}_{33} \\ 0 \end{Bmatrix} \tag{20.153}$$

is admissible if $\dot{\lambda}_1 \geq 0$ and $a_{21}\dot{\lambda}_1 - b_2 \geq 0$, which is equivalent to

$$\dot{e}_{22} - \dot{e}_{33} \geq 0 \tag{20.154}$$
$$2\dot{e}_{11} - \dot{e}_{22} - \dot{e}_{33} \leq 0 \tag{20.155}$$

(b) A complementary situation arises when only surface 2 remains active. The admissibility conditions for this solution are

$$2\dot{e}_{22} - \dot{e}_{11} - \dot{e}_{33} \leq 0 \tag{20.156}$$
$$\dot{e}_{11} - \dot{e}_{33} \geq 0 \tag{20.157}$$

3. *No surface remains active*, i.e. the stress point moves to the elastic domain and no plastic flow takes place. Both rates $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are zero. The solution $\dot{\boldsymbol{\lambda}} = \mathbf{0}$ is admissible if $\boldsymbol{A}\dot{\boldsymbol{\lambda}} - \boldsymbol{b} = -\boldsymbol{b} \geq \mathbf{0}$, i.e. if

$$\dot{e}_{22} - \dot{e}_{33} \leq 0 \tag{20.158}$$
$$\dot{e}_{11} - \dot{e}_{33} \leq 0 \tag{20.159}$$

Comparing the admissibility conditions for the previous solutions, we realize that they decompose the strain rate space into four sectors with disjoint interiors. This confirms that the response to any prescribed strain rate is unique, as was expected owing to the *positive definiteness* of matrix $\boldsymbol{A}$.

The results have a clear geometrical interpretation. The four sectors are plotted in the deviatoric section of the principal stress space in Figure 20.7. Note that, in the present case, the solution depends only on the deviatoric part of the strain rate, $\dot{\boldsymbol{e}}$. This is natural since the yield criterion is pressure-insensitive and, if an associated flow rule is used, the volumetric response is purely elastic and completely decomposed from the deviatoric response. Moreover, only the normal components of $\dot{\boldsymbol{e}}$ with respect

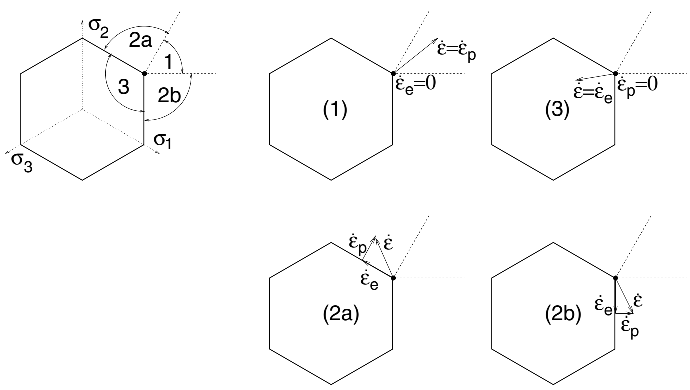

Figure 20.7 Plastic flow at a corner of Tresca yield surface

to the principal stress coordinates have an effect on the solution. These components are linked by the condition $\dot{e}_{11}+\dot{e}_{22}+\dot{e}_{33} = 0$, and so only two of them are independent and the presentation in the two-dimensional deviatoric plane covers all the possible situations. The deviatoric part of the elastic stress rate is colinear with the deviatoric part of the strain rate because the elastic law has the simple form $\dot{\boldsymbol{s}} = 2G\dot{\boldsymbol{e}}$. We can, therefore, interpret the vectors (starting from the corner of the yield surface in Figure 20.7) alternatively as representing the strain rates or the stress rates.

Now we can discuss the four types of solutions. In case (1), the strain rate (and also the elastic stress rate) points to the sector between the normals to the straight segments of the yield surface. It is therefore possible to express the strain rate as a linear combination of the normal vectors with nonnegative coefficients. The total strain rate is equal to the plastic strain rate and the elastic strain rate vanishes. Consequently, the stress rate must also vanish and the stress point remains at the corner. In case (3), the elastic stress rate points to the interior of the elastic domain and is identical to the actual stress rate. No plastic flow takes place and the process is fully elastic. Finally, in case (2), the strain rate points to one of the remaining two sectors. Its orthogonal projection on the yield surface is located either on the segment where $f_1 = 0$ or on that where $f_2 = 0$. The projection on the yield surface represents the elastic part of the strain rate and the component normal to the yield surface is the plastic part. The stress point moves from the corner along one or the other straight segment of the yield surface.

20.4.2 Hardening Plasticity

The next challenge is an extension of the previous results to hardening plasticity. A crucial point to note is that, during hardening, two cases can occur: (1) either the auxiliary surfaces expand or move simultaneously, driven by the same hardening variables; or (2) each of the surfaces is associated with its own hardening variables that only grow if the surface is active. The former case corresponds to plasticity theory with a single non-smooth surface, in which the auxiliary surfaces only describe individual smooth portions of the basic surface. An example is Tresca or Mohr–Coulomb plasticity with isotropic hardening or softening. The latter case is referred to as *multi-surface plasticity*, or plasticity theory with multiple yield conditions. Here, each surface represents a different mechanism of plastic flow, independent of the others and controled by its own internal variables (though couplings can be introduced, too). This class of models includes plasticity-based microplane models (Carol and Bažant, 1997). Another example is a model based on a combination of the Drucker–Prager yield surface (representing failure under predominantly compressive stresses) with the Rankine surface (representing tensile cracking), recently proposed for concrete by Feenstra and de Borst (1996).

Multiple yield surfaces are also used to model nonlinear kinematic hardening by a set of nested surfaces, each of which only exhibits linear hardening (Iwan, 1967; Mróz, 1967, 1969). However, this is a somewhat different problem because the individual surfaces do not intersect each other, and all the active surfaces have a common tangent at the current stress point.

Let us start with the *non-smooth single-surface plasticity*, only for convenience described by a set of auxiliary functions

$$f_I(\boldsymbol{\sigma}, \kappa) = F_I(\boldsymbol{\sigma}) - \bar{h}(\kappa), \qquad I = 1, 2, \ldots N \tag{20.160}$$

The time derivatives of these functions are

$$\dot{f}_I(\boldsymbol{\sigma},\kappa)=\frac{\partial F_I}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}-\frac{\mathrm{d}\bar{h}}{\mathrm{d}\kappa}\dot{\kappa}=\boldsymbol{f}_{I,\sigma}:\dot{\boldsymbol{\sigma}}-\bar{H}\dot{\kappa},\qquad I=1,2,\dots N \tag{20.161}$$

We will need an expression for the rate of the hardening parameter, $\dot{\kappa}$, in terms of the plastic multiplier rates, $\dot{\lambda}_I$ $(I=1,2,\dots N)$, that appear in the associated flow rule (20.122). If we decided to use the strain hardening model, the rate of the hardening parameter

$$\dot{\kappa}=\sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_p:\dot{\boldsymbol{\varepsilon}}_p}=\sqrt{\frac{2}{3}\sum_{I,J=1}^{N}\dot{\lambda}_I\dot{\lambda}_J\boldsymbol{f}_{I,\sigma}:\boldsymbol{f}_{J,\sigma}} \tag{20.162}$$

would be given by an expression that is nonlinear in terms of the plastic multiplier rates. This would certainly complicate the solution procedure. A more convenient choice is the work hardening hypothesis analogous to (20.8), which leads to a linear expression

$$\dot{\kappa}=\frac{\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}_p}{h(\kappa)}=\sum_{I=1}^{N}\frac{\boldsymbol{\sigma}:\boldsymbol{f}_{I,\sigma}}{h(\kappa)}\dot{\lambda}_I=\sum_{I=1}^{N}k_I\dot{\lambda}_I \tag{20.163}$$

where $k_I=\boldsymbol{\sigma}:\boldsymbol{f}_{I,\sigma}/h(\kappa)$.

Substituting (20.122), (20.163) and the elastic stress-strain law into (20.161), we obtain

$$\begin{aligned}\dot{f}_I&=\boldsymbol{f}_{I,\sigma}:\boldsymbol{D}_e:\left(\dot{\boldsymbol{\varepsilon}}-\sum_{J=1}^{M}\boldsymbol{f}_{J,\sigma}\dot{\lambda}_J\right)-\bar{H}\sum_{J=1}^{M}k_J\dot{\lambda}_J\\&=\boldsymbol{f}_{I,\sigma}:\boldsymbol{D}_e:\dot{\boldsymbol{\varepsilon}}-\sum_{J=1}^{M}(\boldsymbol{f}_{I,\sigma}:\boldsymbol{D}_e:\boldsymbol{f}_{J,\sigma}+\bar{H}k_J)\dot{\lambda}_J,\qquad I=1,2,\dots M\end{aligned} \tag{20.164}$$

We took into account only the active surfaces $1,2,\dots M$ because $\dot{\lambda}_I=0$ for $I=M+1,M+2,\dots N$.

The consistency conditions (20.128) again lead to a linear complementarity problem of the form (20.131) but this time with matrix $\boldsymbol{A}$ having components $a_{IJ}=\boldsymbol{f}_{I,\sigma}:\boldsymbol{D}_e:\boldsymbol{f}_{J,\sigma}+\bar{H}k_J$ which could be nonsymmetric. However, for Tresca, and Mohr–Coloumb yield criteria, the product $\sigma:f_{J,\sigma}$ is the same for all the active surfaces J, and so all coefficients $K_J(J=1,2,\dots M)$ have the same value and symmetry preserved. The postulate of maximum plastic dissipation extended to hardening materials naturally leads to a modified form of the hardening law, $\dot{\kappa}=\sum\dot{\lambda}_I$, for which the matrix $\boldsymbol{A}$ is always symmetric; see Simo and Hughes (1998) and Problem 23.9.

Let us shift attention to *multi-surface plasticity*. Each yield surface now represents a specific failure mechanism, and the evolution of the surface is controled by a hardening variable associated only with this mechanism. For the sake of simplicity, we only consider isotropic hardening of each individual loading surface. However, note that as each surface evolves at its own pace, the resulting change of the composite surface is not isotropic. The individual loading surfaces are described by

$$f_I(\boldsymbol{\sigma},\kappa_I)\equiv F_I(\boldsymbol{\sigma})-\bar{h}_I(\kappa_I)=0,\qquad I=1,2,\dots N \tag{20.165}$$

where the hardening functions $\bar{h}_I$ $(I=1,2,\dots N)$ are in general different. We can start from the assumption that the inactive surfaces remain stationary, i.e. they exhibit neither hardening nor softening. Consequently, the hardening variable associated with an inactive surface should remain constant. It is therefore natural to relate the rate of hardening variable associated with a certain surface only to the part of the plastic strain rate (or of the plastic work rate) due to yielding on that surface. This leads us to the evolution equation for the hardening variables,

$$\dot{\kappa}_I = \|\dot{\lambda}_I \boldsymbol{f}_{I,\sigma}\| = \dot{\lambda}_I \|\boldsymbol{f}_{I,\sigma}\| \equiv \dot{\lambda}_I k_I(\boldsymbol{\sigma}), \qquad I = 1, 2, \ldots N \tag{20.166}$$

for strain hardening, and

$$\dot{\kappa}_I = \frac{1}{\bar{h}(\kappa_I)} \boldsymbol{\sigma} : \dot{\lambda}_I \boldsymbol{f}_{I,\sigma} = \dot{\lambda}_I \frac{\boldsymbol{\sigma} : \boldsymbol{f}_{I,\sigma}}{\bar{h}(\kappa_I)} \equiv \dot{\lambda}_I k_I(\boldsymbol{\sigma}, \kappa_I), \qquad I = 1, 2, \ldots N \tag{20.167}$$

for work hardening. Evaluating the time derivatives of the yield functions,

$$\dot{f}_I = \boldsymbol{f}_{I,\sigma} : \dot{\boldsymbol{\sigma}} - \frac{\mathrm{d}\bar{h}_I}{\mathrm{d}\kappa_I} \dot{\kappa}_I = \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \dot{\boldsymbol{\varepsilon}} - \sum_{J=1}^{M} (\boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{J,\sigma}) \dot{\lambda}_J - \bar{H}_I k_I \dot{\lambda}_I \tag{20.168}$$

we find that the coefficients of matrix $\boldsymbol{A}$ from (20.129) are now given by

$$a_{IJ} = \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{J,\sigma} + \bar{H}_I k_I \delta_{IJ}, \qquad I, J = 1, 2, \ldots M \tag{20.169}$$

where δ_{IJ} is Kronecker delta but no summation over repeated subscripts is implied. Again, we only took into account the active surfaces $1, 2, \ldots M$ because $\dot{\lambda}_I = 0$ for $I = M+1, M+2, \ldots N$. Now the matrix $\boldsymbol{A}$ is again symmetric and, compared to the case of perfect plasticity with a_{IJ} given by (20.130), only the diagonal terms are modified. For an arbitrary vector $\boldsymbol{x}$ we have

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} &= \sum_{I,J=1}^{M} x_I \boldsymbol{f}_{I,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{J,\sigma} x_J + \sum_{I=1}^{M} \bar{H}_I k_I x_I^2 \\ &= \left(\sum_{I=1}^{M} x_I \boldsymbol{f}_{I,\sigma} \right) : \boldsymbol{D}_e : \left(\sum_{I=1}^{M} x_I \boldsymbol{f}_{I,\sigma} \right) + \sum_{I=1}^{M} \bar{H}_I k_I x_I^2 \end{aligned} \tag{20.170}$$

If all the plastic moduli $\bar{H}_i$ are positive, matrix $\boldsymbol{A}$ with components given by (20.169) is positive definite even for linearly dependent gradients $\boldsymbol{f}_{I,\sigma}$, and the corresponding linear complementarity problem (20.131) has a unique solution. With zero or negative plastic moduli, uniqueness might be lost.

Example 20.10: Find the condition of local uniqueness for a model combining the Drucker–Prager criterion with the Rankine criterion. Assume that the evolution of each yield surface is described by a work hardening law with no interaction between the two hardening mechanisms.

Solution: The given model uses two yield functions[1]

$$f_1(\boldsymbol{\sigma}, \kappa_1) = \alpha I_1(\boldsymbol{\sigma}) + \sqrt{J_2(\boldsymbol{\sigma})} - \bar{h}_1(\kappa_1) \tag{20.171}$$

[1] Strictly speaking, the Rankine surface is also a composite surface, because it has singularities at points where two or three principal stresses are equal. Our aim here is to illustrate the methodology, and so we focus on the intersection of the Drucker–Prager surface with the Rankine surface (at a regular point of the Rankine surface), and we leave a detailed analysis of other possible singular points to the reader as an exercise; see Problem 20.8.

$$f_2(\boldsymbol{\sigma}, \kappa_2) = \sigma_1(\boldsymbol{\sigma}) - \bar{h}_2(\kappa_2) \tag{20.172}$$

Their gradients with respect to the stress tensor are easily computed as

$$\boldsymbol{f}_{1,\sigma} = \alpha \frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{1}{2\sqrt{J_2}} \frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \alpha \boldsymbol{\delta} + \frac{1}{2\sqrt{J_2}} \boldsymbol{s} \tag{20.173}$$

$$\boldsymbol{f}_{2,\sigma} = \frac{\partial \sigma_1}{\partial \boldsymbol{\sigma}} = \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 \tag{20.174}$$

where $\boldsymbol{n}_1$ is a unit vector in the direction of maximum principal stress. Now we evaluate

$$\boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} = 3K\alpha\boldsymbol{\delta} + \frac{G}{\sqrt{J_2}} \boldsymbol{s} \tag{20.175}$$

$$\boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} = \boldsymbol{D}_e : (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) \tag{20.176}$$

$$\boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} = 9K\alpha^2 + G \tag{20.177}$$

$$\boldsymbol{f}_{2,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} = 3K\alpha(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) : \boldsymbol{\delta} + \frac{G}{\sqrt{J_2}} (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) : \boldsymbol{s} = 3K\alpha + \frac{Gs_1}{\sqrt{J_2}} \tag{20.178}$$

$$\boldsymbol{f}_{2,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} = (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) : \boldsymbol{D}_e : (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) = D^e_{1111} = K + \frac{4}{3}G \tag{20.179}$$

$$\boldsymbol{\sigma} : \boldsymbol{f}_{1,\sigma} = \alpha I_1 + \frac{1}{2\sqrt{J_2}} 2J_2 = \alpha I_1 + \sqrt{J_2} \tag{20.180}$$

$$\boldsymbol{\sigma} : \boldsymbol{f}_{2,\sigma} = \boldsymbol{\sigma} : (\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) = \sigma_1 \tag{20.181}$$

where s_1 is the maximum principal deviatoric stress. If surface 1 is active, we have $\alpha I_1 + \sqrt{J_2} - \bar{h}_1 = 0$, and so

$$k_1 = \frac{\boldsymbol{\sigma} : \boldsymbol{f}_{1,\sigma}}{\bar{h}_1} = \frac{\alpha I_1 + \sqrt{J_2}}{\bar{h}_1} = 1 \tag{20.182}$$

A similar result, $k_2 = 1$, is obtained also for surface 2. We can see that, in the present case, the hardening variables κ_I are identical with the plastic multipliers λ_I.

The existence and uniqueness of the solution is guaranteed if all the principal minors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} + \bar{H}_1 k_1 & \boldsymbol{f}_{1,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} \\ \boldsymbol{f}_{2,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{1,\sigma} & \boldsymbol{f}_{2,\sigma} : \boldsymbol{D}_e : \boldsymbol{f}_{2,\sigma} + \bar{H}_2 k_2 \end{bmatrix} = \begin{bmatrix} 9K\alpha^2 + G + \bar{H}_1 & 3K\alpha + Gs_1/\sqrt{J_2} \\ 3K\alpha + Gs_1/\sqrt{J_2} & K + 4G/3 + \bar{H}_2 \end{bmatrix} \tag{20.183}$$

are positive. The principal minors of a 2×2 matrix are the diagonal coefficients and the determinant. This leads to three inequalities

$$9K\alpha^2 + G + \bar{H}_1 > 0 \tag{20.184}$$

$$K + \tfrac{4}{3}G + \bar{H}_2 > 0 \tag{20.185}$$

$$(9K\alpha^2 + G + \bar{H}_1)(K + \tfrac{4}{3}G + \bar{H}_2) > \left(3K\alpha + \frac{Gs_1}{\sqrt{J_2}}\right)^2 \tag{20.186}$$

The first two conditions set a lower bound on $\bar{H}_1$ and $\bar{H}_2$, respectively. They correspond to the yield modes in which only one of the surfaces remains active. For given material parameters, the bounds can easily be evaluated. The last condition is quadratic and contains both $\bar{H}_1$ and $\bar{H}_2$. It corresponds to the mode in which both surfaces evolve simultaneously. In addition to material parameters, the right-hand side contains also the first principal deviatoric stress, s_1, and the second deviatoric invariant, J_2. To guarantee uniqueness for all possible situations, we have to consider the most unfavorable case, in which the ratio $s_1/\sqrt{J_2}$ is maximized (because then the right-hand side of (20.186) has the largest possible value and the resulting restriction on $\bar{H}_1$ and $\bar{H}_2$ is the most severe). Fortunately, the task of maximizing $s_1/\sqrt{J_2}$ is easy if we use the Haigh–Westergaard coordinates. According to (D.71) in Appendix D, we have $s_1 = \sqrt{2/3}\,\rho\cos\theta$, where $\rho = \sqrt{2J_2}$, and so $s_1/\sqrt{J_2} = \sqrt{4/3}\cos\theta$ attains the largest possible value for $\theta = 0$, in which case $s_1/\sqrt{J_2} = \sqrt{4/3}$. Condition (20.186) can now be written exclusively in terms of material parameters;

$$\left(9K\alpha^2 + G + \bar{H}_1\right)\left(K + \frac{4}{3}G + \bar{H}_2\right) > \left(3K\alpha + \frac{2G}{\sqrt{3}}\right)^2 \tag{20.187}$$

In the $(\bar{H}_1, \bar{H}_2)$-plane, the boundary of the region in which (20.187) holds is a hyperbola centered at $\bar{H}_1 = -(9K\alpha^2 + G)$ and $\bar{H}_2 = -(K + 4G/3)$.

For illustration we plot conditions (20.184), (20.185) and (20.187) in the plane of the plastic moduli normalized by Young's modulus, with a specific choice of Poisson's ratio, $\nu = 0.2$, and the friction coefficient, $\alpha = 0.21$ (these values are representative of concrete). Figure 20.8 indicates that the third condition, which takes into account simultaneous yielding, reduces the region of admissible plastic moduli as compared to the case in which each yield surface is considered separately. For example, if we consider only the Rankine surface, the plastic modulus $\bar{H}_2$ must be larger than $-1.111E$. If we combine the Rankine criterion with softening and the Drucker–Prager criterion with perfect plasticity ($\bar{H}_1 = 0$), the critical value of the softening modulus $\bar{H}_2$ becomes $-0.027E$. Additional restrictions could be obtained by analyzing the edges and the vertex of the Rankine surface (i.e. the cases for which two or three principal stresses are equal); see Problem 20.8.

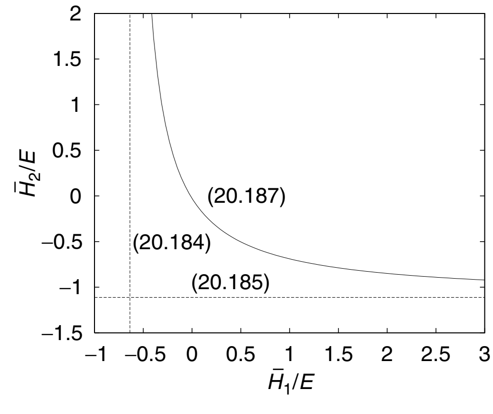

Figure 20.8 Restrictions on plastic moduli for the model combining the Rankine and Drucker–Prager surfaces

20.5 ANISOTROPIC YIELD CRITERIA

Some materials (e.g. oriented fiber-matrix composites or wood) have a strongly anisotropic structure, which must be reflected not only by the elastic moduli, but also by the yield condition. The description of the material is no longer invariant with respect to arbitrary rotations of the coordinate system, and so the yield criterion should be expressed in terms of the stress components with respect to a certain (fixed) coordinate system rather than in terms of the invariants of the stress tensor.

20.5.1 Hill Criterion

We will restrict our attention to *orthotropic materials*, i.e. to materials with three mutually orthogonal planes of symmetry. It is natural to relate the material description to the axes of orthotropy x_1, x_2 and x_3, defined as the intersections of the planes of symmetry. The stress components with respect to the axes of orthotropy are denoted as σ_{11}, σ_{12}, etc. The orthotropic yield condition

$$\left(\frac{\sigma_{22}-\sigma_{33}}{k_{11}}\right)^2+\left(\frac{\sigma_{33}-\sigma_{11}}{k_{22}}\right)^2+\left(\frac{\sigma_{11}-\sigma_{22}}{k_{33}}\right)^2 +\left(\frac{\sigma_{23}}{k_{23}}\right)^2+\left(\frac{\sigma_{31}}{k_{31}}\right)^2+\left(\frac{\sigma_{12}}{k_{12}}\right)^2-1=0 \tag{20.188}$$

came first, proposed by Hill (1947). It is a straightforward generalization of the von Mises condition; cf. expression (D.40) for the invariant J_2. As for the von Mises condition, Hill's condition is insensitive to volumetric stress and possesses tension-compression symmetry.

The six material parameters k_{11}, k_{22}, k_{33}, k_{12}, k_{23} and k_{31} can be related to the yield stresses in tension and compression along the three axes of orthotropy, σ^0_{11}, σ^0_{22} and σ^0_{33}, and to the yield stresses in shear in the three planes of symmetry, τ^0_{12}, τ^0_{23} and τ^0_{31}. For uniaxial tension along axis x_1,

$$\left(\frac{\sigma^0_{11}}{k_{22}}\right)^2+\left(\frac{\sigma^0_{11}}{k_{33}}\right)^2=1 \tag{20.189}$$

and two similar conditions can be obtained by cyclic permutation of the subscripts. Solving for the material parameters, we identify

$$k_{11}=\sqrt{2}\left(\frac{1}{(\sigma^0_{22})^2}+\frac{1}{(\sigma^0_{33})^2}-\frac{1}{(\sigma^0_{11})^2}\right)^{-1/2} \tag{20.190}$$

and analogous expressions can be written for k_{22} and k_{33}.

For pure shear in the plane x_1–x_2 we get

$$\left(\frac{\tau^0_{12}}{k_{12}}\right)^2=1 \tag{20.191}$$

from which

$$k_{12}=\tau^0_{12} \tag{20.192}$$

and similar relations $k_{23}=\tau^0_{23}$ and $k_{31}=\tau^0_{31}$ hold for the other two planes of symmetry. If the material is isotropic, we have $\sigma^0_{11}=\sigma^0_{22}=\sigma^0_{33}=\sqrt{3}\tau_0$ and $\tau^0_{12}=\tau^0_{23}=\tau^0_{31}=\tau_0$. The reader can verify that (20.188) is then equivalent to the von Mises yield condition.

20.5.2 Hoffman Criterion

The Hill criterion does not allow modeling of materials with different values of the yield stress in tension and in compression. Hoffman (1967) added a linear combination of the normal stresses σ_{11}, σ_{22} and σ_{33}, to make the yield surface nonsymmetric with respect to the origin. To facilitate parameter identification, we write the Hoffman criterion in the form

$$\left(\frac{\sigma_{22}-\sigma_{33}-c_{22}+c_{33}}{k_{11}}\right)^2+\left(\frac{\sigma_{33}-\sigma_{11}-c_{33}+c_{11}}{k_{22}}\right)^2 \\ +\left(\frac{\sigma_{11}-\sigma_{22}-c_{11}+c_{22}}{k_{33}}\right)^2+\left(\frac{\sigma_{23}}{k_{23}}\right)^2 \\ +\left(\frac{\sigma_{31}}{k_{31}}\right)^2+\left(\frac{\sigma_{12}}{k_{12}}\right)^2-1=0 \tag{20.193}$$

so that the new parameters c_{11}, c_{22}, and c_{33} correspond to the shifted center of the yield surface. They can be easily identified as

$$c_{11}=\frac{\sigma_{11}^{0+}-\sigma_{11}^{0-}}{2}, \qquad c_{22}=\frac{\sigma_{22}^{0+}-\sigma_{22}^{0-}}{2}, \qquad c_{33}=\frac{\sigma_{33}^{0+}-\sigma_{33}^{0-}}{2} \tag{20.194}$$

where σ_{11}^{0+} is the tensile yield stress in direction 1, $-\sigma_{11}^{0-}<0$ is the compressive yield stress in direction 1, etc. Equations of the type (20.190) and (20.192) remain valid if we replace σ_{11}^{0} by $(\sigma_{11}^{0+}+\sigma_{11}^{0-})/2$, etc.

20.5.3 Tsai–Wu Criterion

The most general quadratic yield condition was proposed by Tsai and Wu (1971). They defined the yield function by

$$f(\boldsymbol{\sigma})=\boldsymbol{f}:\boldsymbol{\sigma}+\boldsymbol{\sigma}:\boldsymbol{F}:\boldsymbol{\sigma}-1 \tag{20.195}$$

where $\boldsymbol{f}$ is a symmetric second-order tensor and $\boldsymbol{F}$ is a fourth-order tensor with both major and minor symmetry. In engineering notation, $\boldsymbol{f}$ can be represented by a column matrix with six coefficients and $\boldsymbol{F}$ by a symmetric 6×6 matrix with 21 independent coefficients (for a general anisotropic material). If the material is orthotropic, the number of independent material parameters can be reduced because the yield function must be invariant with respect to reflections about the axes of symmetry. For example, after reflection about the plane (x_1, x_2), σ_{23} and σ_{31} change sign but σ_{12} does not, and so the coefficients multiplying the products $\sigma_{23}\sigma_{12}$ and $\sigma_{31}\sigma_{12}$ must vanish. By similar arguments we arrive at the Tsai–Wu criterion for orthotropic materials,

$$\left\{\begin{matrix}f_1\\f_2\\f_3\end{matrix}\right\}^T\left\{\begin{matrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\end{matrix}\right\}+\left\{\begin{matrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{matrix}\right\}^T\begin{bmatrix}F_{11}&F_{12}&F_{13}&0&0&0\\F_{12}&F_{22}&F_{23}&0&0&0\\F_{13}&F_{23}&F_{33}&0&0&0\\0&0&0&F_{44}&0&0\\0&0&0&0&F_{55}&0\\0&0&0&0&0&F_{66}\end{bmatrix}\left\{\begin{matrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{matrix}\right\}=1 \tag{20.196}$$

This criterion contains 12 material parameters.

The Tsai–Wu criterion can be reduced to the Hoffman and Hill criteria. In other words, conditions (20.193) and (20.188) can be written in the form (20.196), with some internal dependencies among the material parameters. Verification of this fact is left to the reader as an exercise.

20.5.4 Comparison

In summary, we have three orthotropic yield criteria with an increasing level of complexity. They are due to Hill, Hoffman, and Tsai and Wu, and they use (respectively) 6, 9 and 12 parameters. Hill's criterion is insensitive to the volumetric stress and gives the same yield stresses in uniaxial tension and compression. Hoffman's criterion allows for different behaviors in tension and in compression, but is still insensitive to the volumetric part of the stress tensor. The Tsai–Wu criterion is the most general quadratic yield condition, and appears to be the most rational one. Unfortunately, it requires the determination of 12 material parameters, which is often impossible due to the scarcity of experimental data. In its most general form (20.195), this criterion is applicable to materials with general anisotropy, but the number of parameters then increases to 27.

Example 20.11: Reduce the orthotropic yield conditions to a plane-stress situation (in one of the planes of material symmetry) and present them graphically. Discuss the differences.

Solution: Let us start from the simplest case. *Hill's criterion* (20.188) reduces under plane-stress conditions to

$$\left(\frac{\sigma_{22}}{k_{11}}\right)^2 + \left(\frac{\sigma_{11}}{k_{22}}\right)^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{k_{33}}\right)^2 + \left(\frac{\sigma_{12}}{k_{12}}\right)^2 - 1 = 0 \qquad (20.197)$$

This equation describes an ellipsoid in the $(\sigma_{11}, \sigma_{22}, \sigma_{12})$-space. The ellipsoid has its center at the origin and one of its principal axes coincides with the axis σ_{12}. The intersection with the plane $(\sigma_{11}, \sigma_{22})$ is an ellipse with principal axes in a general position. The criterion (20.197) has four parameters. Parameter k_{12} has the meaning of the yield stress in shear. Formula (20.190) for parameter k_{11} and analogous formulae for k_{22} and k_{33} contain the yield stress for tension and compression in the out-of-plane direction x_3. This does not appear to be logical from the physical point of view. For identifying the parameters to be used in the criterion for plane stress, it is more reasonable to ignore the test results in the out-of-plane direction and supplement the tensile yield stresses in directions x_1 and x_2 by a yield stress corresponding to another loading path, e.g. by the yield stress σ^0_{et} corresponding to equibiaxial tension with $\sigma_{11} = \sigma_{22}$. Substituting this into (20.197) we obtain the relation

$$\left(\frac{\sigma^0_{et}}{k_{11}}\right)^2 + \left(\frac{\sigma^0_{et}}{k_{22}}\right)^2 = 1 \qquad (20.198)$$

and so the parameters k_{11}, k_{22} and k_{33} can be evaluated from formulae analogous to (20.190), with σ^0_{33} replaced by σ^0_{et}.

Hoffman's criterion (20.193) reduces under plane-stress conditions to

$$\left(\frac{\sigma_{22}-c_{22}}{k_{11}}\right)^2 + \left(\frac{\sigma_{11}-c_{11}}{k_{22}}\right)^2 + \left(\frac{\sigma_{11}-\sigma_{22}-c_{11}+c_{22}}{k_{33}}\right)^2 + \left(\frac{\sigma_{12}}{k_{12}}\right)^2 - 1 = 0 \quad (20.199)$$

This is still an ellipsoid in the space $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ with one principal axis parallel to σ_{12} but its center is now shifted to the point $(c_{11}, c_{22}, 0)$. The criterion contains six material parameters. Instead of using the formulae that depend on the yield stresses in the out-of-plane direction, it would again be more appropriate to identify the parameters from six experiments performed under plane stress. Parameter k_{12} is equal to the yield stress in shear, and the remaining parameters can be related to the four yield stresses in tension and in compression along axes x_1 and x_2, supplemented by one additional yield stress, e.g. that obtained under equibiaxial tension.

The *Tsai–Wu criterion* (20.196) reduced to a plane-stress situation reads

$$F_{11}\sigma_{11}^2 + 2F_{12}\sigma_{11}\sigma_{22} + F_{22}\sigma_{22}^2 + F_{66}\sigma_{12}^2 + f_1\sigma_{11} + f_2\sigma_{22} - 1 = 0 \qquad (20.200)$$

Note also that it also involves six materials parameters, the same as the reduced Hoffman criterion. If $F_{11}F_{22} > F_{12}^2$ and $F_{66} > 0$, (20.200) describes an ellipsoid and the elastic domain is convex. In this case, criteria (20.200) and (20.199) are fully equivalent. The only difference is that they are expressed in terms of different material parameters.

PROBLEMS

Problem 20.1: Show that, for a purely deviatoric flow under monotonic uniaxial loading, the rate of cumulative plastic strain $\dot{\bar{\varepsilon}}_p$ defined by (20.5) is equal to the usual plastic strain rate $\dot{\varepsilon}_{11}^p$.

Problem 20.2: Plot the uniaxial stress-strain curve that would correspond to linear work hardening, defined by a hardening law of the form $\sigma_Y = \sigma_0 + cW_p$, where c is a constant and W_p is the plastic work (20.7). Is the stress-strain curve linear?

Problem 20.3: Consider a strain-hardening or strain-softening Rankine material with an associated flow rule. Derive the condition of uniqueness in terms of the plastic modulus.

Problem 20.4*: Consider an elastoplastic material with linear strain softening subjected to uniaxial tension. Combining the basic equations $\varepsilon = \varepsilon_e + \varepsilon_p$, $\varepsilon_e = \sigma/E$, and $\dot{\varepsilon}_p = \dot{\sigma}/H$, show that the rate form of the stress-strain law reads

$$\dot{\sigma} = \frac{EH}{E+H}\dot{\varepsilon} \qquad (20.201)$$

The uniqueness of response of the model is preserved only if $E+H > 0$. This condition is more restrictive than condition (20.40) derived in Example 20.2. Explain why.

Problem 20.5: Consider a material with isochoric plastic flow and with purely kinematic hardening. Find the relation between the Melan–Prager hardening parameter, $\bar{H}_k$, and the plastic modulus, H, defined as the proportionality factor between the stress rate and the plastic strain rate under uniaxial tension.

Problem 20.6: Figure 20.3(b) shows the evolution of the von Mises yield surface during kinematic hardening under uniaxial stress. The center of the ellipse moves along the axis σ_1, but the flow direction has a nonzero component $\dot{\varepsilon}_2$. Is this in contradiction with the Melan–Prager kinematic hardening rule?

Problem 20.7*: Consider an elastic domain with a non-smooth boundary, described by $f_I(\boldsymbol{\sigma}) < 0$ $(I = 1, 2)$, and a nonassociated flow rule $\dot{\boldsymbol{\varepsilon}} = \dot{\lambda}\boldsymbol{g}_\sigma$ with a *smooth* plastic potential. (a) Formulate the consistency condition(s). (b) Find the expression for the rate of plastic multiplier in a situation in which $f_1 = 0$ and $f_2 = 0$. (c) Discuss the local uniqueness of the model.

Problem 20.8: For the composite yield surface combining the Rankine and Drucker–Prager conditions, explore singular points that were not considered in Example 20.10. Set up the conditions of local uniqueness and plot them in the plane of plastic moduli.

Problem 20.9: Explore the singular points of the composite yield surface combining the Rankine and Drucker–Prager conditions for plane stress problems. Set up the conditions of local uniqueness and plot them in the plane of plastic moduli. Make a comparison to the three-dimensional case.

Problem 20.10: In the proof of the global uniqueness theorem for perfectly plastic materials given in Section 16.3 it was tacitly assumed that the yield surface is smooth. Show that the theorem remains valid for yield surfaces with corners.

Problem 20.11: Write the Hill criterion (20.188) in the form of the Tsai–Wu criterion (20.196). What are the internal dependencies among the 12 Tsai–Wu parameters?

Problem 20.12: Write the Hoffman criterion (20.193) in the form of the Tsai–Wu criterion (20.196). What are the internal dependencies among the 12 Tsai–Wu parameters?

Problem 20.13: An orthotropic material is called transversely isotropic if its properties are invariant with respect to arbitrary rotations about one axis of orthotropy, say x_1. Write the Tsai–Wu criterion with a reduced number of independent parameters, suitable for transversely isotropic materials.