8.7 Associated and Non-associated Flow Rules

Recall the Levy-Mises flow rule, Eqn. 8.4.3,

$$d\varepsilon_{ij}^{p} = d\lambda s_{ij} \tag{8.7.1}$$

The plastic multiplier can be determined from the hardening rule. Given the hardening rule one can more generally, instead of the particular flow rule 8.7.1, write

$$d\varepsilon_{ij}^{p} = d\lambda G_{ij}\,, \tag{8.7.2}$$

where G_{ij} is some function of the stresses and perhaps other quantities, for example the hardening parameters. It is symmetric because the strains are symmetric.

A wide class of material behaviour (perhaps all that one would realistically be interested in) can be modelled using the general form

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}\,. \tag{8.7.3}$$

Here, g is a scalar function which, when differentiated with respect to the stresses, gives the plastic strains. It is called the **plastic potential**. The flow rule 8.7.3 is called a **non-associated flow rule**.

Consider now the sub-class of materials whose plastic potential *is* the yield function, $g = f$:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}\,. \tag{8.7.4}$$

This flow rule is called an **associated flow-rule**, because the flow rule is associated with a particular yield criterion.

8.7.1 Associated Flow Rules

The yield surface $f(\sigma_{ij}) = 0$ is displayed in Fig 8.7.1. The axes of principal stress and principal plastic strain are also shown; the material being isotropic, these are taken to be coincident. The normal to the yield surface is in the direction $\partial f / \sigma_{ij}$ and so the associated flow rule 8.7.4 can be interpreted as saying that *the plastic strain increment vector is normal to the yield surface*, as indicated in the figure. This is called the **normality rule**.

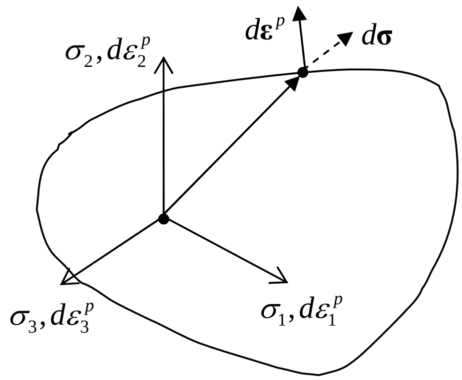

Figure 8.7.1: Yield surface

The normality rule has been confirmed by many experiments on metals. However, it is found to be seriously in error for soils and rocks, where, for example, it overestimates plastic volume expansion. For these materials, one must use a non-associative flow-rule.

Next, the Tresca and Von Mises yield criteria will be discussed. First note that, to make the differentiation easier, the associated flow-rule 8.7.4 can be expressed in terms of principal stresses as

$$d\varepsilon_i^p = d\lambda \frac{\partial f}{\partial \sigma_i}. \tag{8.7.5}$$

Tresca

Taking $\sigma_1 > \sigma_2 > \sigma_3$, the Tresca yield criterion is

$$f = \frac{\sigma_1 - \sigma_3}{2} - k \tag{8.7.6}$$

One has

$$\frac{\partial f}{\partial \sigma_1} = +\frac{1}{2}, \quad \frac{\partial f}{\partial \sigma_2} = 0, \quad \frac{\partial f}{\partial \sigma_3} = -\frac{1}{2} \tag{8.7.7}$$

so, from 8.7.5, the flow-rule associated with the Tresca criterion is

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} +\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}. \tag{8.7.8}$$

This is the flow-rule of Eqns. 8.4.33. The plastic strain increment is illustrated in Fig. 8.7.2 (see Fig. 8.3.9). All plastic deformation occurs in the $1-3$ plane. Note that 8.7.8 is independent of stress.

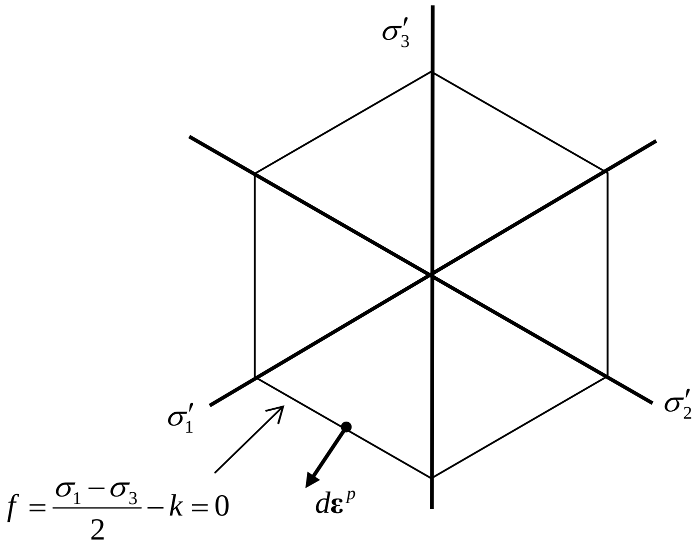

Figure 8.7.2: The plastic strain increment vector and the Tresca criterion in the π-plane (for the associated flow-rule)

Von Mises

The Von Mises yield criterion is $f = J_2 - k^2 = 0$. With

$$\frac{\partial J_2}{\partial \sigma_1} = \frac{\partial}{\partial \sigma_1}\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{2}{3}\left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right] \quad (8.7.9)$$

one has

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} \frac{2}{3}\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right) \\ \frac{2}{3}\left(\sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_3)\right) \\ \frac{2}{3}\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right) \end{bmatrix}. \quad (8.7.10)$$

This are none other than the Levy-Mises flow rule 8.4.6[1].

The associative flow-rule is very appealing, connecting as it does the yield surface to the flow-rule. Many attempts have been made over the years to justify this rule, both mathematically and physically. However, it should be noted that the associative flow-rule is not a law of nature by any means. It is simply very convenient. That said, it

---

[1] note that if one were to use the alternative but equivalent expression $f = \sqrt{J_2} - k = 0$, one would have a $1/2\sqrt{J_2}$ term common to all three principal strain increments, which could be "absorbed" into the $d\lambda$ giving the same flow-rule 8.7.10

does agree with experimental observations of many plastically deforming materials, particularly metals.

In order to put the notion of associative flow-rules on a sounder footing, one can define more clearly the type of material for which the associative flow-rule applies; this is tied closely to the notion of stable and unstable materials.

8.7.2 Drucker's Postulate

Stress Cycles

First, consider the one-dimensional loading of a hardening material. The material may have undergone any type of deformation (e.g. elastic or plastic) and is now subjected to the stress σ^*, point A in Fig. 8.7.3. An *additional* load is now applied to the material, bringing it to the current yield stress σ at point B (if σ^* is below the yield stress) and then plastically (greatly exaggerated in the figure) through the infinitesimal increment $d\sigma$ to point C. It is conventional to call these additional loads the **external agency**. The external agency is then removed, bringing the stress back to σ^* and point D. The material is said to have undergone a **stress cycle**.

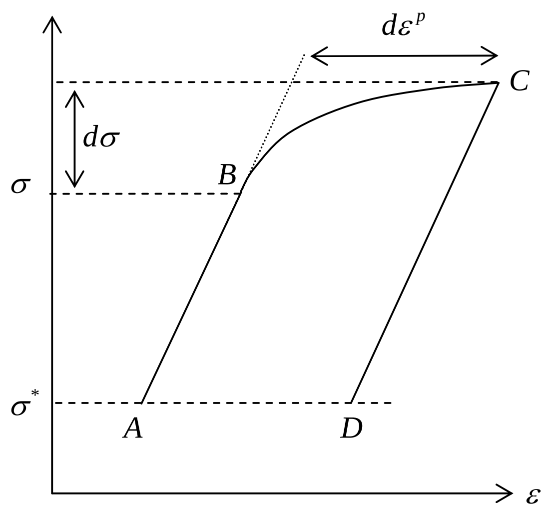

Figure 8.7.3: A stress cycle for a hardening material

Consider now a softening material, Fig. 8.7.4. The external agency first brings the material to the current yield stress σ at point B. To reach point C, the loads must be reduced. This cannot be achieved with a stress (force) control experiment, since a reduction in stress at B will induce elastic unloading towards A. A strain (displacement) control must be used, in which case the stress required to induce the (plastic) strain will be seen to drop to $\sigma + d\sigma$ ($d\sigma < 0$) at C. The stress cycle is completed by unloading from C to D.

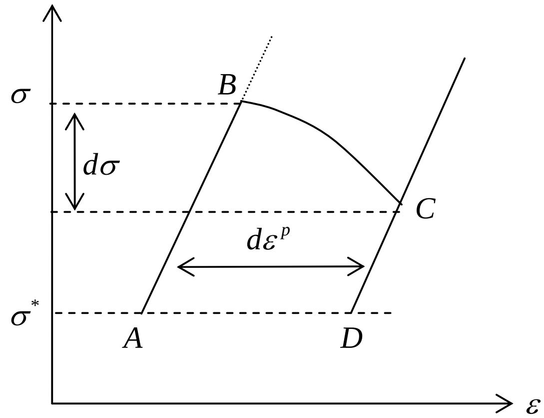

Figure 8.7.4: A stress cycle for a softening material

Suppose now that $\sigma = \sigma^*$, so the material is at point B, on the yield surface, before action by the external agency. It is now not possible for the material to undergo a stress cycle, since the stress cannot be increased. This provides a means of distinguishing between strain hardening and softening materials:

Strain-hardening … Material can always undergo a stress-cycle
Strain-softening … Material cannot always undergo a stress-cycle

Drucker's Postulate

The following statements define a **stable material**: (these statements are also known as **Drucker's postulate**):

(1) Positive work is done by the external agency during the application of the loads
(2) The net work performed by the external agency over a stress cycle is nonnegative

By this definition, it is clear that a strain hardening material is stable (and satisfies Drucker's postulates). For example, considering plastic deformation ($\sigma = \sigma^*$ in the above), the work done during an increment in stress is $d\sigma d\varepsilon$. The work done by the external agency is the area shaded in Fig. 8.7.5a and is clearly positive (note that the work referred to here is not the *total* work, $\int_{\varepsilon}^{\varepsilon+d\varepsilon} \sigma d\varepsilon$, but only that part which is done by the external agency[2]). Similarly, the net work over a stress cycle will be positive.

On the other hand, note that plastic loading of a softening (or perfectly plastic) material results in a non-positive work, Fig. 8.7.5b.

[2] the laws of thermodynamics insist that the total work is positive (or zero) in a complete cycle.

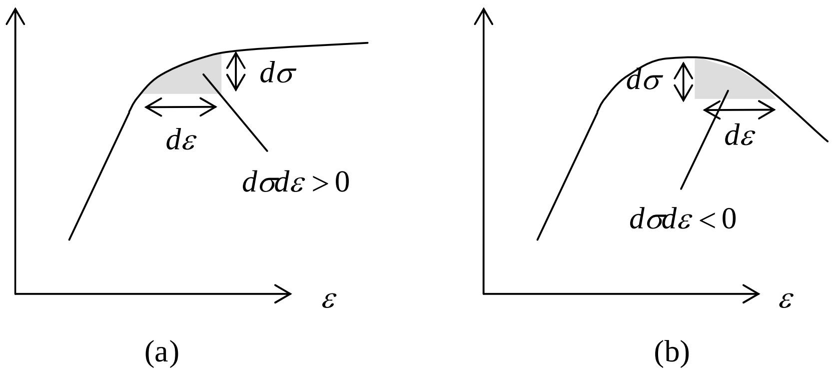

Figure 8.7.5: Stable (a) and unstable (b) stress-strain curves

The work done (per unit volume) by the additional loads during a stress cycle A-B-C-D is given by:

$$W = \int_{A-B-C-D} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon \tag{8.7.11}$$

This is the shaded work in Fig. 8.7.6. Writing $d\varepsilon = d\varepsilon^e + d\varepsilon^p$ and noting that the elastic work is recovered, i.e. the net work due to the elastic strains is zero, this work is due to the plastic strains,

$$W = \int_{B-C} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon^p \tag{8.7.12}$$

With $d\sigma$ infinitesimal, this equals

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \tag{8.7.13}$$

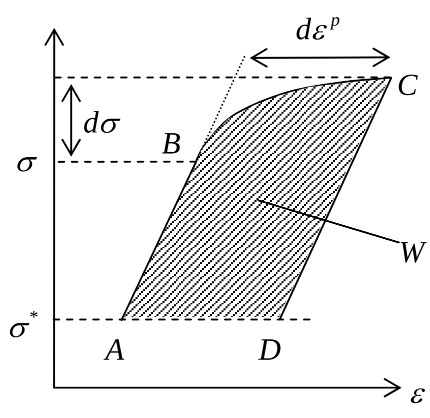

Figure 8.7.6: Work *W* done during a stress cycle of a strain-hardening material

The requirement (2) of a stable material is that this work be non-negative,

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \geq 0 \tag{8.7.14}$$

Making $\sigma - \sigma^* >> d\sigma$, this reads

$$\left(\sigma - \sigma^*\right) d\varepsilon^p \geq 0 \tag{8.7.15}$$

On the other hand, making $\sigma = \sigma^*$, it reads

$$d\sigma d\varepsilon^p \geq 0 \tag{8.7.16}$$

The three dimensional case is illustrated in Fig. 8.7.7, for which one has

$$\left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p \geq 0, \quad d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \tag{8.7.17}$$

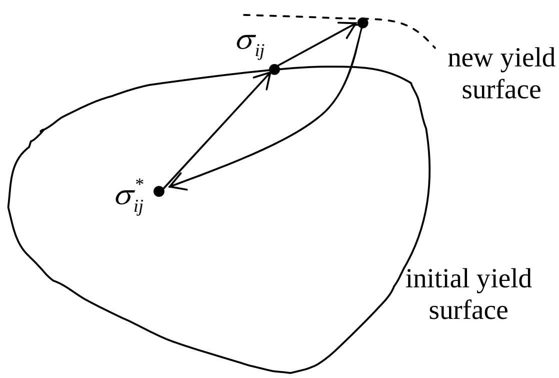

Figure 8.7.7: Stresses during a loading/unloading cycle

8.7.3 Consequences of the Drucker's Postulate

The criteria that a material be stable have very interesting consequences.

Normality

In terms of vectors in principal stress (plastic strain increment) space, Fig. 8.7.8, Eqn. 8.7.17 reads

$$\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\right) \cdot d\boldsymbol{\varepsilon}^p \geq 0 \tag{8.7.18}$$

These vectors are shown with the solid lines in Fig. 8.7.8. Since the dot product is non-negative, the angle between the vectors $\boldsymbol{\sigma} - \boldsymbol{\sigma}^*$ and $d\boldsymbol{\varepsilon}^p$ (with their starting points coincident) must be less than 90°. This implies that the plastic strain increment vector must be normal to the yield surface since, if it were not, an initial stress state $\boldsymbol{\sigma}^*$ could be found for which the angle was greater than 90° (as with the dotted vectors in Fig. 8.7.8). Thus a consequence of a material satisfying the stability requirements is that the **normality rule** holds, i.e. the flow rule is associative, Eqn. 8.7.4.

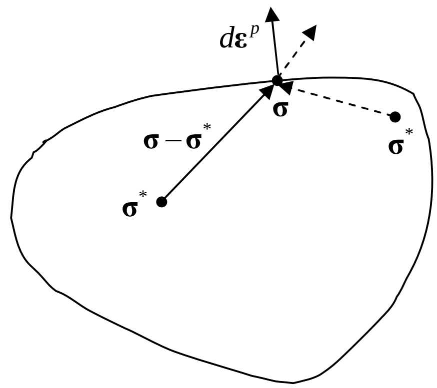

Figure 8.7.8: Normality of the plastic strain increment vector

When the yield surface has sharp corners, as with the Tresca criterion, it can be shown that the plastic strain increment vector must lie within the cone bounded by the normals on either side of the corner, as illustrated in Fig. 8.7.9.

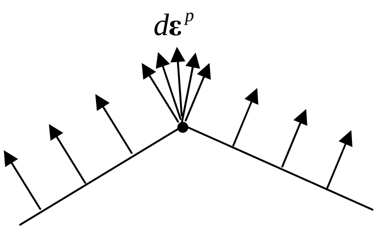

Figure 8.7.9: The plastic strain increment vector for sharp corners

Convexity

Using the same arguments, one cannot have a yield surface like the one shown in Fig. 8.7.10. In other words, the yield surface is **convex**: the entire elastic region lies to one side of the tangent plane to the yield surface[3].

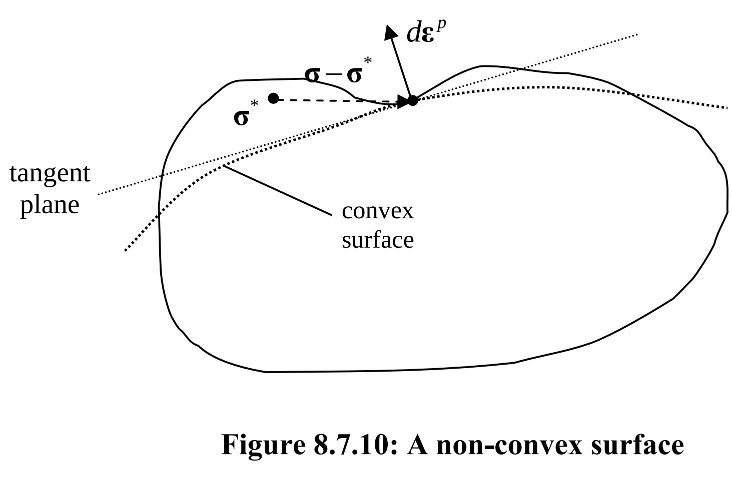

Figure 8.7.10: A non-convex surface

[3] note that when the plastic deformation affects the elastic response of the material, it can be shown that the stability postulate again ensures normality, but that the convexity does not necessarily hold

In summary then, Drucker's Postulate, which is satisfied by a stable, strain-hardening material, implies normality (associative flow rule) and convexity[4].

8.7.4 The Principle of Maximum Plastic Dissipation

The rate form of Eqn. 8.7.18 is

$$\left(\sigma_{ij} - \sigma_{ij}^{*}\right)\dot{\varepsilon}_{ij}^{p} \geq 0 \qquad (8.7.19)$$

The quantity $\sigma_{ij}\dot{\varepsilon}_{ij}^{p}$ is called the **plastic dissipation**, and is a measure of the *rate* at which energy is being dissipated as deformation proceeds.

Eqn. 8.7.19 can be written as

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq \sigma_{ij}^{*}\dot{\varepsilon}_{ij}^{p} \quad \text{or} \quad \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}^{p} \geq \boldsymbol{\sigma}^{*}\cdot\dot{\boldsymbol{\varepsilon}}^{p} \qquad (8.7.20)$$

and in this form is known as the **principle of maximum plastic dissipation**: of all possible stress states σ_{ij}^{*} (within or on the yield surface), the one which arises is that which requires the maximum plastic work.

Although the principle of maximum plastic dissipation was "derived" from Drucker's postulate in the above, it is more general, holding also for the case of perfectly plastic and softening materials. To see this, disregard stress cycles and consider a stress state σ^{*} which is at or below the current (yield) stress σ, and apply a strain $d\varepsilon > 0$. For a perfectly plastic material, $\sigma - \sigma^{*} \geq 0$ and $d\varepsilon = d\varepsilon^{p} > 0$. For a softening material, again $\sigma - \sigma^{*} \geq 0$ and $d\varepsilon^{e} < 0$, $d\varepsilon^{p} > d\varepsilon > 0$.

It follows that the normality rule and convexity hold also for the perfectly plastic and softening materials which satisfy the principle of maximum plastic dissipation.

In summary:

Drucker's postulate leads to the Principle of maximum plastic dissipation
For hardening materials
    Principle of maximum plastic dissipation leads to Drucker's postulate
For softening materials
    Principle of maximum plastic dissipation does not lead to Drucker's postulate

Finally, note that, for many materials, hardening and softening, a non-associative flow rule is required, as in Eqn. 8.7.3. Here, the plastic strain increment is no longer normal to the yield surface and the principle of maximum plastic dissipation does not hold in general. In this case, when there is hardening, i.e. the stress increment is directed out from the yield surface, it is easy to see that one can have $d\sigma_{ij}d\varepsilon_{ij}^{p} < 0$, Fig. 8.7.11, contradicting the stability postulate (1),. With hardening, there is no

[4] it also ensures the uniqueness of solution to the boundary value elastoplastic problem

obvious instability, and so it could be argued that the use of the term "stability" in Drucker's postulate is inappropriate.

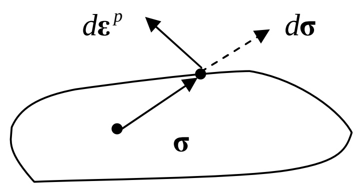

Figure 8.7.11: plastic strain increment vector not normal to the yield surface; non-associated flow-rule

8.7.5 Problems

1. Derive the flow-rule associated with the Drucker-Prager yield criterion $f = \alpha I_1 + \sqrt{J_2} - k$

2. Derive the flow-rule associated with the Mohr-Coulomb yield criterion, i.e. with $\sigma_1 > \sigma_2 > \sigma_3$,
$$\frac{\alpha\sigma_1 - \sigma_3}{2} = k$$
Here,
$$\alpha = \frac{1+\sin\phi}{1-\sin\phi} > 1, \qquad 0 < \phi < \frac{\pi}{2}$$
Evaluate the volumetric plastic strain increment, that is
$$\frac{\Delta V^p}{\Delta V} = d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p,$$
and hence show that the model predicts dilatancy (expansion).

3. Consider the plastic potential
$$g = \frac{\beta\sigma_1 - \sigma_3}{2} - k$$
Derive the non-associative flow-rule corresponding to this potential. Hence show that compaction of material can be modelled by choosing an appropriate value of β.