Chapter 6
Thermoelasticity

1 Introduction

When thermal energy is added to an elastic material it expands. For the simple unidimensional case of a bar of length L, initially at uniform temperature T_0 which is then heated to a nonuniform temperature T and thus grows in length by an amount ΔL, the relative uni-axial stretching due to thermal expansion is

$$\frac{\Delta L}{L} = \epsilon = \alpha(T - T_0)$$

where ϵ is the strain and α is the thermal expansion coefficient. For an isotropic cube of side L the (normal) thermoelastic strains are

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(T - T_0)$$

It is conventional but not necessary to take $T_0 = 0$.

Since the heated region is joined to, and constrained by rigid surroundings, it can not expand freely but becomes subjected to compressive stresses. At the same time the colder portion is subjected to the pull exerted by of the adjacent hot portion and it is thus under tension.

Although Hooke's law is still applicable, due account must be taken of the additional stresses created by thermal expansion.

2 Governing Equations of Thermoelasticity for an Isotropic Solid

The governing equations for the isotropic thermoelastic solid include the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \sigma_{ij,j} + X_i = 0$$

where $i, j = 1, 2, 3$, the generalized thermoelastic stress-strain relations

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(T - T_0) = \lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} - \beta\delta_{ij}\theta$$

where $\theta = T - T_0$ is the excess temperature distribution, and $\beta = \alpha E/(1 - 2\nu)$ where α is the thermal expansion coefficient.

Expressed as strain-stress relationships the above are

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\mu\mu}\delta_{ij} + \alpha\theta\delta_{ij}$$

The small displacement strain-displacement relations are, as before

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Finally, the compatibility equations must also be satisfied.

The temperature distribution θ must be determined by solving the energy conservation equation

$$\frac{dU}{dt} = T\frac{\partial S}{\partial t} + \frac{1}{\rho}\sigma_{ij}V_{ij}$$

where U is the internal energy, S the entropy and

$$V_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

is the rate of deformation tensor where v_i is the velocity.

One can show that the following from the differential thermal energy balance equation can be derived from the above

$$\frac{\partial H}{\partial t} + \theta\beta_{ij}\frac{\partial \epsilon_{ij}}{\partial t} = \nabla \cdot (k\nabla\theta) + r$$

where H is the enthalpy, β_{ij} are experimentally determined numerical coefficients and r is the rate of internal energy generation.

The energy equation above must be solved subject to suitable boundary and initial conditions in order to determine the temperature field θ.

For steady state conditions in a medium of constant conductivity and without internal heat generation

$$\nabla^2\theta = 0$$

i.e. solutions to steady state heat conduction problems are harmonic functions.

In uncoupled, quasi-static thermoelastic theory, the mechanical coupling terms in the energy and the heat conduction equations are neglected. Therefore, the heat conduction problem and the thermoelastic deformation problem are handled separately.

By substituting the generalized thermoelastic stress-strain relations and the small displacement strain-displacement relations into the equilibrium equation one obtains the generalized Navier's equation

$$Gu_{i,\mu\mu} + (\lambda + G)u_{\mu,\mu i} + X_i - \beta\theta_{,i} = 0$$

The three thermomechanical equilibrium equations together with the energy equations and the six stress-strain relations constitute a set of ten equations for the ten unknowns u_i, τ_{ij} and θ. One can show this system is complete, yields an unique solution under suitable boundary conditions and the resulting strain satisfies the compatibility relations.

3 Displacement Potential and Stress Functions

Goodier introduced the displacement potential function ϕ as

$$\mathbf{u} = \nabla\phi = u_i = \frac{\partial \phi}{\partial x_i}$$

this, when substituted into the generalized Navier equation and integrated yields

$$\phi_{,\mu\mu} = \frac{1}{\lambda + 2G}(P + \beta\theta)$$

where P is the potential for the assumed conservative body forces (i.e. $\mathbf{X} = -\nabla P$).

The solution of the above is the sum of a particular solution and the complementary solution of Laplace's equation ($\nabla^2\phi = 0$).

For plane strain conditions, on a $x - y$ plane in rectangular Cartesian coordinates, combination of the equilibrium equations and the compatibility condition yields

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \frac{\beta}{1-\nu}\nabla^2\theta$$

Introducing the stress function Φ defined by

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} + \beta\theta$$

$$\sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} + \beta\theta$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y}$$

yields

$$\nabla^4\Phi = -\frac{\alpha E}{1-\nu}\nabla^2\theta$$

4 Thermal Stresses in a Thin Plate

Consider an infinitely long plate of very small thickness and width $2c$. Let the long direction be aligned with the x axis and the width with y. Assume that $T_0 = 0$ and that the temperature in the plate is only a function of y, (i.e. $\theta = T(y)$). What would be the thermoelastic states of strain and stress resulting from this temperature field?

The answer is obtained using the principle of superposition. First, one must determine the amount of compressive stress that would have to be applied to keep the plate from straining altogether in the longitudinal (x) direction.

From the above, the required stress would be

$$\sigma'_x = -\alpha E T(y)$$

Since one is interested in the thermal stress in an expanding plate, to the above stress one must superimpose the stress generated in the plate when a uniformly distributed tensile force of magnitude

$$\frac{1}{2c}\int_{y=-c}^{y=+c} \alpha E T(y) dy$$

is applied at the $x \to \pm\infty$ boundaries.

Therefore, the actual **thermal stress** in the plate is

$$\sigma_x = \frac{1}{2c}\int_{y=-c}^{y=+c} \alpha E T(y) dy - \alpha E T(y)$$

Assume now that $T(y)$ is quadratic in y,

$$T(y) = T_{y=0}(1 - \frac{y^2}{c^2})$$

I.e. the center of the plate is at temperature $T_{y=0}$ while the edges $y = \pm c$ are at 0. Substituting this into the expression for σ_x gives

$$\sigma_x = \frac{2}{3}\alpha E T_{y=0} - \alpha E T_{y=0}(1 - \frac{y^2}{c^2})$$

Clearly, the stress is quadratic in y. The maximum compressive stress is at $y = 0$ and it is equal to $\sigma_{x,y=0} = -\frac{1}{3}\alpha E T_{y=0}$, while the maximum tensile stress is at $y = \pm c$ and it is $\sigma_{x,y=\pm c} = \frac{2}{3}\alpha E T_{y=0}$. The stress is zero at $y = \pm c/\sqrt{3}$.

5 Thermal Stress in Disks and Cylinders

Consider a thin disk (radius b) with a hole of radius a at the center. Assume the temperature in the disk $\theta = T(r)$ is only a function of the radial position r measured from the center of the hole.

If plane stress conditions are assumed, mechanical equilibrium requires

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\phi}{r} = 0$$

where r and ϕ are the radial and azimuthal directions, respectively.

The strain-displacement relations are

$$\epsilon_r = \frac{du}{dr}$$
$$\epsilon_\phi = \frac{u}{r}$$

where u is the radial displacement.

Finally, for linear thermoelastic material the stress-strain relations are

$$\sigma_r = \frac{E}{1-\nu^2}[(\epsilon_r + \nu\epsilon_\phi) - (1+\nu)\alpha T]$$

$$\sigma_\phi = \frac{E}{1-\nu^2}[(\epsilon_\phi + \nu\epsilon_r) - (1+\nu)\alpha T]$$

Combination of the strain-displacement relations with the above and substitution into the mechanical equilibrium equation yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\alpha\frac{dT}{dr}$$

with the general solution

$$u = (1+\nu)\alpha\frac{1}{r}\int_a^r Trdr + C_1\frac{r}{2} + C_2\frac{1}{r}$$

where C_1, C_2 are constants. The associated stresses are

$$\sigma_r = -\frac{\alpha E}{r^2}\int_a^r Trdr + \frac{E}{1-\nu}\frac{C_1}{2} - \frac{E}{1+\nu}\frac{C_2}{r^2}$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\int_a^r Trdr - \alpha ET + \frac{E}{1-\nu}\frac{C_1}{2} + \frac{E}{1+\nu}\frac{C_2}{r^2}$$

Since no radial stresses act at the inner and outer the outer radius of the disk ($\sigma_r(a) = \sigma_r(b) = 0$),

$$C_1 = \frac{2(1-\nu)\alpha}{b^2 - a^2}\int_a^b Trdr$$
$$C_2 = \frac{(1+\nu)\alpha a^2}{b^2 - a^2}\int_a^b Trdr$$

and the axial strain is

$$\epsilon_z = (1+\nu)\alpha T - \frac{2\nu\alpha}{b^2-a^2}\int_a^b Trdr$$

If plane strain conditions are assumed instead (a good approximation in the case of a tall hollow cylinder with its bases restrained from movement along the axial direction), the corresponding results are, for the displacement

$$u = \frac{\alpha}{r}\frac{1+\nu}{1-\nu}[\int_a^r Trdr + \frac{(1-2\nu)r^2+a^2}{b^2-a^2}\int_a^b Trdr]$$

and for the associated stresses are

$$\sigma_r = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-\int_a^r Trdr + \frac{r^2-a^2}{b^2-a^2}\int_a^b Trdr]$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-Tr^2 + \int_a^r Trdr + \frac{r^2+a^2}{b^2-a^2}\int_a^b Trdr]$$

and

$$\sigma_z = \alpha E\frac{1}{1-\nu}[-T + \frac{2\nu}{b^2-a^2}\int_a^b Trdr]$$

The solution to the case of a tall hollow tube unrestrained from movement in the axial direction is given by

$$u = \frac{\alpha}{r}\frac{1}{1-\nu}[(1+\nu)\int_a^r Trdr + \frac{(1-3\nu)r^2+(1+\nu)a^2}{b^2-a^2}\int_a^b Trdr]$$

the associated stresses are

$$\sigma_r = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-\int_a^r Trdr + \frac{r^2-a^2}{b^2-a^2}\int_a^b Trdr]$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-Tr^2 + \int_a^r Trdr + \frac{r^2+a^2}{b^2-a^2}\int_a^b Trdr]$$

and

$$\sigma_z = \frac{\alpha E}{1-\nu}[-T + \frac{2}{b^2-a^2}\int_a^b Trdr]$$

and the longitudinal strain is

$$\epsilon_z = \frac{2\alpha}{b^2-a^2}\int_a^b Trdr$$

Specifically, for a thin disk with radial steady state temperature distribution

$$T(r) = T_b - (T_b - T_a)\frac{\ln(b/r)}{\ln(b/a)}$$

where $T_a = T(a), T_b = T(b)$. the stresses are

$$\sigma_r = \frac{1}{2}\alpha E(T_b - T_a)[\frac{1-(a/r)^2}{1-(a/b)^2} - \frac{\ln(r/a)}{\ln(b/a)}]$$

$$\sigma_\phi = \frac{1}{2}\alpha E(T_b - T_a)[\frac{1+(a/r)^2}{1-(a/b)^2} - \frac{1+\ln(r/a)}{\ln(b/a)}]$$

with $\sigma_z = 0$. The corresponding stresses for the long hollow cylinder are obtained dividing the above by $1-\nu$. but in this case with

$$\sigma_z = \frac{\alpha E(T_b - T_a)}{2(1-\nu)}[\frac{2}{1-(a/b)^2} - \frac{1+2\ln(r/a)}{\ln(b/a)}]$$

Consider finally the specific example of quenching a long free cylinder, initially at a uniform temperature $T(r) = T_0$ by maintaining its surface temperature at zero ($T(r=b) = 0$). The solution of the homogeneous linear transient 1D heat conduction problem is (see for example "Conduction of Heat in Solids", 2nd ed, by Carslaw and Jaeger, Clarendon, Oxford, 1959, p. 199):

$$T(r) = T_0 \sum_{n=1}^{\infty} \frac{2}{\beta_n J_1(\beta_n)} J_0(\beta_n \frac{r}{b}) e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

where κ is the thermal diffusivity, J_0 and J_1 are the Bessel functions of first kind, of orders zero and one, respectively and β_n are the eigenvalues of the problem, which are the roots of

$$J_0(\beta_n) = 0$$

Substituting the expression for $T(r)$ into the stress equations one obtains

$$\sigma_r(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{1}{\beta_n^2} - \frac{1}{\beta_n^2}\frac{b}{r}\frac{J_1(\beta_n(r/b))}{J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

$$\sigma_\phi(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{1}{\beta_n^2} + \frac{1}{\beta_n^2}\frac{b}{r}\frac{J_1(\beta_n(r/b))}{J_1(\beta_n)} - \frac{J_0(\beta_n(r/b))}{\beta_n J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

and

$$\sigma_z(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{2}{\beta_n^2} - \frac{J_0(\beta_n(r/b))}{\beta_n J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

If one is interested only in the maximum value of the stresses (which occur when $t \approx 0$ at the surface), the results are

$$\sigma_r(b) = 0$$

$$\sigma_\theta = \sigma_z = \frac{\alpha E T_0}{1 - \nu}$$

I.e. the surface of the cylinder is under circumferential (hoop) and axial tensions of equal magnitudes. The (cold) surface layers of the cylinder want to contract but are prevented from doing so by the (still hot) core. If instead of quenching, a cold cylinder is heated, the initial stress state at the surface is compressive.

6 Thermal Stresses in a Sphere

Consider a sphere of radius b in which the temperature is only function of r. The differential mechanical equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_t) = 0$$

where σ_r and σ_t are, respectively, the radial and tangential stress.

The stress-strain relations are:

$$\epsilon_r = \alpha T + \frac{1}{E}(\sigma_r - 2\nu\sigma_t)$$

and

$$\epsilon_t = \alpha T + \frac{1}{E}[\sigma_t - \nu(\sigma_r + \sigma_t)]$$

Finally, the displacement-strain relationships are

$$\epsilon_r = \frac{du}{dr}$$

and

$$\epsilon_t = \frac{u}{r}$$

The solution in this case is

$$u(r) = \frac{1 + \nu}{1 - \nu}\alpha\frac{1}{r^2}\int_0^r T r'^2 dr'$$

$$\sigma_r(r) = \frac{2\alpha E}{1-\nu}[\frac{1}{b^3}\int_0^b Tr^2 dr - \frac{1}{r^3}\int_0^r Tr'^2 dr']$$

and

$$\sigma_t(r) = \frac{\alpha E}{1-\nu}[\frac{2}{b^3}\int_0^b Tr^2 dr + \frac{1}{r^3}\int_0^r Tr'^2 dr' - T]$$

If $T(r)$ is known, stresses are readily computed.

For instance, if a cold solid sphere, initially at T_0, is heated by maintaining its surface at temperature T_1, the maximum compressive stress (occurring at the surface at the very beginning of the process) is

$$\sigma_r(b) = \sigma_t(b) = -\frac{\alpha E(T_1 - T_0)}{1-\nu}$$