Chapter 3
Stress, Strain, Virtual Power and Conservation Principles

1 Introduction

Stress and strain are key concepts in the analytical characterization of the mechanical state of a solid body. While stress represents internal forces per unit area resulting from loads applied to the body, strain is the resulting relative displacement of points in the body. This chapter formally introduces the notions of stress and strain tensors and it also shows how the mechanical equilibrium equations can be obtained directly from the application of the principle of virtual work. The chapter starts with a review of vectors and tensors.

2 Overview of Vectors and Tensors

Tensors are widely used in engineering analysis to denote physical quantities of interest. This section reviews basic notions of tensor analysis needed in continuum mechanics.

2.1 Notation

Tensors are important in applications because governing equations which have general validity with respect to any frame of reference can be constructed by ensuring that every term in the equation has the same tensor characteristics. Thus tensor characteristic play a role analogous to that of dimensional analysis. Thus, once a physical quantity has been given the characteristic of a tensor then the components of the quantity can be transformed from one coordinate system to another according to the above rules.

In vector and tensor calculus, subscript and superscript index notation is used to denote collections of variables, for instance, the set $x_1, x_2, ..., x_n$ is denoted by $x_i, i = 1, 2, ..., n$ or by $x^i, i = 1, 2, ..., n$. Likewise, the set $y^1, y^2, ..., y^n$ is denoted as $y^j, j = 1, 2, ..., n$. Note that the superscript is just an index, not a power. If a power is meant, the quantity will be enclosed in parenthesis.

The *summation convention* is used to simplify the writing of equations consisting of collection of similar looking terms. Whenever a sum involving two identically indexed variables appears one simply writes a single term using a dummy index and omits the summation sign. For instance

$$a_1x_1 + a_2x_2 + a_3x_3 = \sum_{i=1}^{3} a_ix_i = a_ix_i$$

The summation convention also applies to derivatives, specifically, for a function $f(x_1, x_2, x_3)$ the total differential expressed in terms of the partial derivatives is

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 = \frac{\partial f}{\partial x_i}dx_i$$

A concrete example is provided by the unit vector $\mathbf{u}$ in three dimensional Euclidean space in rectangular Cartesian coordinates. In tensor analysis, components are denoted by indices, so instead of writing x, y, z for the three coordinates in such space one writes $x_1, x_2, x_3 = x_i, i = 1, 2, 3$.

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = u_i\mathbf{e}_i$$

where u_i are the components of $\mathbf{u}$ and $\mathbf{e}_i, i = 1, 2, 3$ are the unit coordinate vectors ($\mathbf{i}, \mathbf{j}, \mathbf{k}$ in rectangular Cartesian coordinates, respectively). The magnitude of $\mathbf{u}$, $u = |\mathbf{u}|$, is given by

$$u = \sqrt{u_iu_i} = \sqrt{u_i^2} = 1$$

Therefore, the dot product of two vectors $\mathbf{a}, \mathbf{b}$ can be expressed as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i = \delta_{ij}a_ib_j$$

The quantity

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is *Kronecker's delta*.

Another example is the differential arc or line element of a curve in space ds, this is

$$ds = \sqrt{\delta_{ij}dx_idx_j}$$

where two summations are involved.

Another example is the determinant of a 3×3 matrix $|a_{ij}|$, this is given as

$$|a_{ij}| = e_{rst}a_{r1}a_{s2}a_{t3}$$

where e_{rst} is the *permutation symbol* defined as

$$e_{rst} = \begin{cases} 1 & \text{when subscripts permute like } 1,2,3 \\ 0 & \text{when any two indices coincide} \\ -1 & \text{otherwise} \end{cases}$$

The permutation symbol and Kronecker's delta are related by

$$e_{ijk}e_{rst} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$$

With the above, the vector product of two vectors can be simply expressed as

$$\mathbf{a} \times \mathbf{b} = e_{ijk}a_j b_k$$

2.2 The Euclidian Metric Tensor

Consider a system of rectangular coordinates x_1, x_2, x_3. Consider also a new system of coordinates u_1, u_2, u_3. The two systems being related by the expressions

$$x_i = x_i(u_1, u_2, u_3)$$

or

$$u_i = u_i(x_1, x_2, x_3)$$

so that to every triplet (x_1, x_2, x_3) there corresponds a triplet (u_1, u_2, u_3).

In any system of coordinates, coordinate curves in space are generated by varying one coordinate while holding the other two constant. If the three coordinate curves resulting from the triplet (u_1, u_2, u_3) are mutually perpendicular at each point P , then the triplet constitutes a system of orthogonal curvilinear coordinates.

If a differential segment of an arbitrary curve in space is associated with differential displacements in the coordinates dx^1, dx^2, dx^3 then it can be expressed as The differential element of arc of a curve in coordinates x_i is

$$ds = \sqrt{\delta_{ij}dx^i dx^j} = \sqrt{dx^i dx^i}$$

But

$$dx^i = \frac{\partial x_i}{\partial u_k}du^k$$

therefore

$$(ds)^2 = dx^i dx^i = \frac{\partial x_i}{\partial u_k}\frac{\partial x_i}{\partial u_m}du^k du^m = g_{km}du^k du^m$$

where the functions

$$g_{km}(u_1, u_2, u_3) = g_{mk}(u_1, u_2, u_3) = \frac{\partial x_i}{\partial u_k}\frac{\partial x_i}{\partial u_m}$$

are the components of the *Euclidian metric tensor* in the coordinate system u_1, u_2, u_3.

2.3 Scalars, Vectors and Tensors

Scalars, vectors and tensors are mathematical entities that are used in applications to represent meaningful physical quantities. Consider two systems of coordinates u^i and u^{i*} which are related by the coordinate transformation rules described above. Physical quantities of interest can be represented in any of these two systems. A *scalar* is an entity consisting of a single component and is represented in terms of u^i by the single component (number) ϕ and in terms of u_{i*} by $\phi*$. If the two numbers are one and the same

$$\phi(u^1, u^2, u^3) = \phi^*(u^{1*}, u^{2*}, u^{3*})$$

A scalar is also considered a tensor of rank or order zero.

If an entity has instead three components in each of the coordinate systems is called a *contravariant vector* or *contravariant tensor of order one* and individual components ξ^i and ξ^{i*} in the two systems are related by

$$\xi^{i*}(u^{1*}, u^{2*}, u^{3*}) = \xi^i(u^1, u^2, u^3)\frac{\partial u^{i*}}{\partial u^i}$$

The use of the index as superscript distinguishes contravariant vectors.

Likewise, if an entity has three components in each of the coordinate systems is called a *covariant vector* or *covariant tensor of rank or order one* and individual components ξ_i and ξ_{i*} in the two systems are related by

$$\xi_{i*}(u^{1*}, u^{2*}, u^{3*}) = \xi_i(u^1, u^2, u^3)\frac{\partial u^i}{\partial u^{i*}}$$

The use of the index as subscript distinguishes contravariant vectors. Covariant and contravariant components are identical in rectangular Cartesian systems of coordinates but they are not in curvilinear coordinates. By convention, only the subscript index notation is used to describe vectors in rectangular Cartesian systems of coordinates.

Now, if an entity has nine components one has *tensor of rank or order two.* There are also contravariant T^{ij} and covariant T_{ij} tensors which transform according to

$$T^{ij*}(u^{1*}, u^{2*}, u^{3*}) = T^{mn}(u^1, u^2, u^3)\frac{\partial u^m}{\partial u^{i*}}\frac{\partial u^n}{\partial u^{j*}}$$

and

$$T_{ij*}(u^{1*}, u^{2*}, u^{3*}) = T_{mn}(u^1, u^2, u^3)\frac{\partial u^{i*}}{\partial u^m}\frac{\partial u^{j*}}{\partial u^n}$$

respectively.

Mixed tensor fields of rank two T^i_j can also be defined as well as tensors of higher ranks.

Again, in rectangular Cartesian systems of coordinates, there is no distinction between contravariant and covariant tensors. By convention only the subscript index notation is used to describe tensors in rectangular Cartesian systems of coordinates.

The Kronecker delta defined before can be regarded as a component of a rank two tensor which turns out to be the Euclidian metric tensor (g^{ij}, g_{ij}, g^i_j), while the permutation symbol can be regarded as a component of a rank three tensor called the *permutation tensor* or the *alternator* ϵ^{ijk}.

It should be noted that given a tensor, others can be generated from it by a process called *contraction* which consists of equating and summing a covariant and a contravariant index of a mixed tensor.

2.4 Algebraic Properties of Second Order Tensors

Recall that tensors, just as vectors can be added (each component of the resulting tensor is the sum of the corresponding components in the original tensors). They can also be multiplied according to the rule

$$C_{iklm} = A_{ik}B_{lm}$$

Also, tensors are symmetric if $A_{ij} = A_{ji}$ and antisymmetric if $A_{ij} = -A_{ji}$.

A vector B_i can be obtained from a tensor T_{ik} and an arbitrary vector A_k by multiplication as follows

$$B_i = T_{ik}A_k$$

The new vector $\mathbf{B}$ has generally different magnitude and direction from $\mathbf{A}$. Now, if $B_i = \lambda A_i$, where λ is a scalar, it is called the *characteristic vector* of T_{ik} and the directions associated with it are called the *characteristic or principal directions* of T_{ik}. The axes determined by the principal directions are called the *principal axes* of T_{ik}. The problem of finding the principal axes of a tensor is called the *reduction of* T_{ik} *to principal axes*. The components of $\mathbf{A}$ determining the principal axes of T_{ik} satisfy the system of equations

$$T_{ik}A_k - \lambda A_i = (T_{ik} - \lambda\delta_{ik})A_k = 0$$

This system has a nontrivial solution only when the determinant

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = \lambda^3 - \lambda^2 I_1 + \lambda I_2 - I_3 = 0$$

where the quantities

$$I_1 = T_{11} + T_{22} + T_{33} = T_{ii}$$

$$I_2 = \begin{vmatrix} T_{22} & T_{32} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{21} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{13} & T_{33} \end{vmatrix}$$

and

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}$$

are called the *invariants* of the tensor T_{ik}.

The equation

$$\lambda^3 - \lambda^2 I_1 + \lambda I_2 - I_3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

is called the *characteristic equation* for the determination of the eigenvalues of a tensor.

2.5 Partial Derivatives in Cartesian Coordinates

In Cartesian coordinates, the partial derivatives of any tensor field are the components of another tensor field. Consider two Cartesian systems of coordinates (x_1, x_2, x_3) and (x_1^*, x_2^*, x_3^*) related by the rule

$$x_i^* = a_{ij} x_j + b_i$$

where a_{ij} and b_i are constants. Let $\xi^{i*}(x_1^*, x_2^*, x_3^*)$ be a contravariant tensor so that

$$\xi^{i*}(x_1^*, x_2^*, x_3^*) = \xi^i(x_1, x_2, x_3) \frac{\partial x_i^*}{\partial x_\alpha}$$

then one has the relationship

$$\frac{\partial \xi^{i*}}{\partial x_j^*} = \frac{\partial \xi^\alpha}{\partial x_\beta} \frac{\partial x_\beta}{\partial x_j^*} \frac{\partial x_i^*}{\partial x_\alpha}$$

i.e. the partial derivatives of ξ transform as a rank two tensor in Cartesian coordinates. This is not the case in curvilinear coordinate systems.

The comma notation is often used to denote partial derivatives. For instance the tensors $\phi_{,i} = \partial\phi/\partial x_i$, $\xi_{i,j} = \partial\xi_i/\partial x_j$ and $\sigma_{ij,k} = \partial\sigma_{ij}/\partial x_k$ are of rank one, two and three, respectively assuming that ϕ, ξ_i and σ_{ij} are tensors of ranks zero, one and two, respectively.

Further, the covariant derivative of the covariant vector ξ_i is defined as

$$\xi_i|_\alpha = \frac{\partial \xi_i}{\partial x^\alpha} - \Gamma_{i\alpha}^\sigma \xi_\sigma$$

and they are the components of a covariant tensor of rank two. Here, the quantity

$$\Gamma_{\alpha\beta}^i = \frac{1}{2} g^{i\sigma} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right)$$

is called the Euclidian Christoffel symbol.

2.6 Characteristics of Tensor Equations

The key property of tensor fields is that if all the components of a tensor vanish in a given coordinate system, they also vanish in all other systems obtainable from the first by admissible transformations. As a consequence, a tensor equation established in one coordinate system will also hold in any other system obtainable from the first by admissible transformations.

For instance, the mass contained inside a given volume V is

$$M = \int\int\int_V \rho_0(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int\int\int_V \rho_0 |\frac{\partial x_i}{\partial x_j^*}| dx_1^* dx_2^* dx_3^*$$

Also the total volume contained inside a closed surface is

$$V = \int\int\int_V dx_1 dx_2 dx_3 = \int\int\int_V |\frac{\partial x_i}{\partial x_j^*}| dx_1^* dx_2^* dx_3^* = \int\int\int_V \sqrt{g} dx_1^* dx_2^* dx_3^*$$

2.7 Geometric Interpretation of Tensor Components

Recall that the set of unit vectors or base vectors, $\mathbf{i}_r$ for $r = 1, 2, 3$ in Euclidean space is a set of linearly independent vectors such that any vector in the space can be generated from them by simple linear combination. Consider an infinitesimal vector $d\mathbf{r} = dx^r \mathbf{i}_r = dx_r \mathbf{i}^r$ connecting two closely space points in space referred to a Cartesian coordinate system. In a new and arbitrary coordinate system $u^i = u^i(x_1, x_2, x_3)$, the same vector is represented as

$$d\mathbf{r} = \mathbf{g}_r du^r = \mathbf{g}^r du_r$$

where $\mathbf{g}_r = (\partial x^s / \partial u^r)\mathbf{i}_s$ is the covariant base vector and $\mathbf{g}^r$ is the contravariant base vector. Moreover,

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i}$$

so that $\mathbf{g}_i$ represents the change in the position vector $\mathbf{r}$ with u^i and points along the tangent to the coordinate curve.

It can be shown that $\mathbf{g}_r \cdot \mathbf{g}_s = g_{rs}$, $\mathbf{g}^r \cdot \mathbf{g}^s = g^{rs}$ and $\mathbf{g}^r \cdot \mathbf{g}_s = g_s^r = \delta_s^r$.

A vector $\mathbf{v}$ can then be expressed

$$\mathbf{v} = v^r \mathbf{g}_r = v_s \mathbf{g}^s$$

and the contravariant components v^r of $\mathbf{v}$ are the components in the direction of the covariant base vectors and vice versa.

Consider two coordinate systems. The associated base vectors are $\mathbf{g}_i, \mathbf{g}^i$ and $\mathbf{g}_i^*, \mathbf{g}^{i*}$. Then, the transformation laws for a vector are

$$v^{i*} = \mathbf{g}^i \cdot \mathbf{g}_m v^m$$

and

$$v^j = \mathbf{g}_i^* \cdot \mathbf{g}^j v^r$$

Likewise, in the case of tensors of rank two the transformation laws are

$$A^{rs*} = \mathbf{g}^{r*} \cdot \mathbf{g}_m \mathbf{g}^{s*} \cdot \mathbf{g}_n A^{mn}$$

and

$$A^{mn} = \mathbf{g}_r^* \cdot \mathbf{g}^m \mathbf{g}_s^* \cdot \mathbf{g}^n A^{mn*}$$

3 Analysis of Stress

3.1 Concept of Stress

When external loads are applied to a solid body, forces are transmitted through body's interior. Stress is a concept used to represent the mechanical interaction across imaginary surfaces in the interior of solid bodies.

Consider a closed surface enclosing an interior region of a solid body. The surface can be characterized by its outward pointing normal vector ν. The material outside the surface exerts a force $\mathbf{F}$ over the adjacent material on the other side of the surface. The *stress vector* $\mathbf{T}$ is the force per unit area and is defined as

$$\mathbf{T} = \frac{d\mathbf{F}}{dS}$$

In a rectangular Cartesian system of coordinates $\mathbf{T}$ has three components, $T_i, i = 1, 2, 3$. Cauchy first pointed out that the force exerted by the material behind the surface on the material outside the surface is equal in magnitude and opposite in sign.

If the region enclosed by the surface has the shape of a cube and a rectangular Cartesian system of coordinates is introduced such that the cube faces are normal to the coordinate axes, there are three components of $\mathbf{T}$ on each of the three positive faces of the cube. These nine numbers are the *stresses* τ_{ij} where the subscript i indicates the plane on which the force acts and the subscript j denotes the direction of action. If $i = j$ one has *normal stresses* and if $i \neq j$ one has shearing stresses. With the above, the stress vector components are expressed as

$$T_i = \nu_j \tau_{ji}$$

Because of Cauchy's idea, the nine components of stress above are necessary and sufficient to characterize the state of stress in a body.

The stresses can be readily represented on a second (primed) rectangular Cartesian system of coordinates according to the following transformation rule

$$\tau'_{km} = \tau_{ji} \frac{\partial x'_k}{\partial x_j} \frac{\partial x'_m}{\partial x_i}$$

3.2 Laws of Motion

As load is applied on a body of volume V, the particles that make up the body are displaced. For any given particle, its position vector is $\mathbf{r}$ and its velocity is $\mathbf{v}$.

The linear momentum of the body is defined as

$$\mathbf{P} = \int_V \mathbf{v}\rho dV$$

and its moment of momentum is defined as

$$\mathbf{H} = \int_V \mathbf{r} \times \mathbf{v}\rho dV$$

where B is the space occupied by the body.

If the total force applied on the body is $\mathbf{F}_T$ and the total applied torque is $\mathbf{L}_T$ the laws of motion are

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_T$$

and

$$\frac{d\mathbf{H}}{dt} = \mathbf{L}_T$$

The forces applied on bodies are of two types: body forces and surface forces. Body forces $\mathbf{X}$ act in the interior of the body while surface forces $\mathbf{T}$ act on surface elements. Gravity is a good example of a body force while stress is an example of surface force. Therefore,

$$\mathbf{F} = \int_V \mathbf{X}dV + \oint_S \mathbf{T}dS$$

3.3 Equilibrium Equations

The equations of equilibrium are simply the statements that no net force and no moment act on a body in a state of mechanical equilibrium. They are easily obtained by carrying out force and moment balances on the cube shaped volume element mentioned above and then taking the limit as the size goes to zero. The results are

$$\frac{\partial \tau_{ij}}{\partial x_j} + X_i = \tau_{ij,j} + X_i = 0$$

for the force equation, and

$$\tau_{ij} = \tau_{ji}$$

for the moment equation.

3.4 Principal Stresses

There are always three perpendicular directions at any point inside a loaded body where the shear stresses vanish. These are called *principal directions* and the planes normal to them are the *principal planes*. The three principal stresses be $\sigma_1, \sigma_2, \sigma_3$ and are the roots of the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma + I_3 = 0$$

where I_1, I_2, I_3 are the *stress tensor invariants* given by

$$I_1 = \tau_{ii}$$

$$I_2 = \frac{1}{2}(\tau_{ii}\tau_{jj} - \tau_{ij}\tau_{ji})$$

$$I_3 = det\tau_{ij}$$

At a point in a loaded body, the mean stress σ_0 is defined as

$$\sigma_0 = \frac{1}{3}\tau_{ii}$$

and the *stress deviation tensor* τ'_{ij} is defined as

$$\tau'_{ij} = \tau_{ij} - \sigma_0\delta_{ij}$$

The invariants of the stress deviation tensor are

$$J_1 = 0$$

$$J_2 = \frac{1}{2}\tau'_{ij}\tau'_{ij} = 3\sigma_0^2 - I_2 = \frac{3}{2}\tau_0^2$$

$$J_3 = \frac{1}{3}\tau'_{ij}\tau'_{jk}\tau'_{ki} = I_3 + J_2\sigma_0 - \sigma_0^3$$

where τ_0 is the *octahedral stress*.

A useful graphical representation of the state of stress at a point can be obtained by drawing the stress Mohr circle.

4 Analysis of Strain

4.1 Concept of Strain

As loads are applied to a body, individual material particles are displaced from their positions. Let the point coordinates before deformation be a^i and x^i after it. An infinitesimal element of arc connecting two adjacent points in the body ds_0 distorts to ds. The difference between the squares of the length elements is given by

$$ds^2 - ds_0^2 = 2E_{ij}da^i da^j$$

or

$$ds^2 - ds_0^2 = 2e_{ij}dx^i dx^j$$

where

$$E_{ij} = \frac{1}{2}(g_{\alpha\beta}\frac{\partial x_\alpha}{\partial a_i}\frac{\partial x_\beta}{\partial a_j} - a_{ij})$$

is the Green-St. Venant (or Lagrangian) strain tensor and

$$e_{ij} = \frac{1}{2}(g_{ij} - a_{\alpha\beta}\frac{\partial a_\alpha}{\partial x_i}\frac{\partial a_\beta}{\partial x_j})$$

is the Almansi (or Eulerian) strain tensor.

One can show, that the necessary and sufficient condition for rigid body motion is the vanishing of the strain tensor.

Since the strain tensors are tensors, they exhibit similar properties to those of the stress tensor. Specifically, one can define strain invariants (the first one, for instance is $e_{ii} = \Delta V/V$ and is called the *dilatation*. Strain deviation tensors can also be defined.

Note that if rectangular Cartesian coordinates are used to describe the deformation $g_{ij} = a_{ij} = \delta_{ij}$. In this case, defining components of the *displacement vector* $\mathbf{u}$ as

$$u_i = x_i - a_i$$

yields

$$E_{ij} = \frac{1}{2}[\frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_\alpha}{\partial a_i}\frac{\partial u_\alpha}{\partial a_j}]$$

and

$$e_{ij} = \frac{1}{2}[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j}]$$

For the important case of *small deformations* (i.e. infinitesimal displacements) the product terms are negligible and one obtains

$$e_{ij} = \epsilon_{ij} = \frac{1}{2}[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}]$$

I.e. for the case of small deformations the Lagrangian and Eulerian strains are the same.

Note that if an element of a body is stretched in the $x-$direction by an amount dx,

$$ds^2 - ds_0^2 = 2e_{xx}(dx)^2$$

i.e. e_{xx} represents an *extension* (change of length per unit length). If instead the element is sheared in the $x - y$ plane, the *shear* is e_{xy}. Therefore e_{ii} are called normal strains and e_{ij} are shearing strains (although engineers sometimes use this name for the quantity $2e_{ij}$). Furthermore, the quantity

$$\omega_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j})$$

is called the *rotation*.

Deformation is assumed to take place without the formation of cracks or voids or interpenetration of materials particles. This requirement is expressed by the *equations of compatibility*. These equations must be fulfilled by the strain components of any admissible deformation field. They are

$$e_{ij,kl} + e_{kl,ij} - e_{ij,jl} - e_{jl,ik} = 0$$

Although the above represent 81 equations, only six turn out to be essential.

A useful graphical representation of the state of strain at a point can be obtained by drawing the strain Mohr circle.

5 Virtual Power

Virtual motions are useful concepts in mechanics of material. They are used both in the analytical formulation of problems and also constitute the foundation of the finite element methodology. Virtual motions are imaginary movements of material points and the method of virtual power consists of determining the associated work or power involved. If the virtual motion of point M is described by the vector $\mathbf{v}$, the associated power is $P(\mathbf{v}(M))$. In this section we show how the static equilibrium equation is readily obtained by applying the principle of virtual power.

The virtual motion can be described with reference to the coordinates of the initial location of point M, M_0 (Lagrangian description) or in terms of the current coordinates

of M (Eulerian description). The total time rate of change of $\mathbf{v}$ in Lagrangian variables is simply $\gamma = d\mathbf{v}/dt = \partial\mathbf{v}/\partial t$ while in the Eulerian description one has

$$\gamma = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_{i,j} v_j$$

where $\nabla \mathbf{v}$ is an important second order tensor, the velocity gradient tensor which can be expressed as

$$\nabla \mathbf{v} = \Omega + \mathbf{D} = \frac{1}{2}(v_{i,j} - v_{j,i}) + \frac{1}{2}(v_{i,j} + v_{j,i})$$

where $\mathbf{D} = D_{ij}$ is the rate of deformation tensor and $\Omega = \Omega_{ij}$ is the rate of rotation tensor.

The fundamental laws of dynamics are embodied in the principle of virtual power. According to it, for a body in mechanical equilibrium, for any virtual motion the virtual power associated with rigid body movement is zero and the virtual power of inertia forces equals the sum of the virtual powers of all internal and external forces.

Consider a material body of volume V and surface S which is subjected to a body force density X_i. Further, let the internal stress field be τ_{ij} and the surface density of cohesive forces T_i. The principle of virtual power is expressed as

$$-\int_V \tau_{ij} D_{ij} dV + \int_V X_i v_i dV + \oint_S T_i v_i dS = \int_V \gamma_i v_i \rho dV$$

Since $T_i = \tau_{ij} n_j$, integration by parts yields

$$\int_V (\tau_{ij,j} + X_i - \rho \gamma_i) v_i dV = 0$$

Since the integrand must vanish, for the special but important case of zero inertia forces one obtains the static equilibrium equation

$$\tau_{ij,j} + X_i = 0$$

As mentioned before, analysis is simpler if the assumption of small displacements and strains can be used. If this is not the case, finite deformation theory must be used to describe the geometry of the deformation.

6 Conservation Principles

Conservation principles are balance statements for physical quantities the amounts of which are conserved in physical processes. Conservation principles are valid regardless the material constitution of the medium in which they apply, therefore, they have a universal character. Mathematical expressions for these conservation principles in the form of differential

equations are readily obtained by performing balances of the conserved quantities over differential volume elements and then taking the limit as the volume shrinks down to zero. In the thermomechanics of solids and fluids the conservation principles most frequently invoked are:

The principle of mass conservation, or equation of continuity, i.e.

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_j}{\partial x_j} = 0$$

where ρ is the local density (mass per unit volume), v_j is the local velocity, x_j are the local Eulerian coordinates and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \rho v_j \frac{\partial}{\partial x_j}$$

is the material or substantial derivative.

The principle of conservation of linear momentum, or equation of motion, i.e.

$$\rho\frac{Dv_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i$$

where σ_{ij} is the stress tensor and X_i is the body force vector. Note this become the standard equilibrium equation under static conditions (i.e. $v_i = 0$).

The principle of balance of angular momentum, or stress tensor symmetry equation, i.e.

$$\sigma_{ij} = \sigma_{ji}$$

The principle of conservation of energy, or first law of thermodynamics, i.e.

$$\rho\frac{Du}{Dt} = -\frac{\partial q_i}{\partial x_i} + r + \sigma_{ij}\frac{\partial v_i}{\partial x_j}$$

where u is the specific internal energy per unit mass, q_i is the heat flux vector, r is the distributed rate of internal energy generation and the last term on the right hand side represents the rate of irreversible degradation of mechanical to thermal energy.

The principle of entropy production or second law of thermodynamics, i.e.

$$\rho\frac{D\eta}{Dt} = -\frac{\partial (q_i/T)}{\partial x_i} + \rho\frac{D\eta_{int}}{Dt}$$

where η is the specific entropy per unit mass, q_i/T is the entropy flux vector (with T being the absolute temperature) and η_{int} is the internal entropy production per unit mass. The last term above equals zero for reversible processes, is greater than zero for irreversible ones and there are no processes in nature for which it is negative. Because of the nature of this last term, the entropy equation is not directly useful in the determination of the various

fields but rather acts as a constraint condition that must be fulfilled by all solutions of the other conservation equations.

All the above balance equations can be represented by a single expression by introducing the generic conserved quantity ψ, its internal supply per unit mass s and the influx of ψ per unit area $-j_i n_i$ where n_i is the normal vector. With the above, the generic conservation equation becomes

$$\rho \frac{D\psi}{Dt} = \rho s - \frac{\partial j_i}{\partial x_i}$$

In the case of a single component system, the equations of continuity, motion and energy constitute a set of five scalar equations involving the unknowns, velocity (three components), temperature, density, stress (six components), energy, heat flux (three components) and entropy; a total of sixteen unknowns. Clearly, the balance equations of thermomechanics constitute a severely underdetermined system and additional equations are required in order to be able to produce well posed problems that can be solved by standard mathematical methods.

The additional equations that must be incorporated in order to solve actual technical problems consist of mathematical descriptions of individual material response or behavior. Such equations are called constitutive equations. These equations, in contrast with the conservation principles, do not have a universal character but rather describe in detail individual material behavior. Fortunately, constitutive equations are available for whole groups of material behavior, namely, elastic, viscoelastic, plastic, creep and viscoplastic and others. In formulating problems in solid mechanics one then proceeds by identifying the appropriate constitutive equations to use, then combines these equations with the balance equations, applies boundary conditions and proceeds to solve the resulting system of equations using analytical or more commonly, numerical solution techniques such as finite element methods.