Governing Equations

3-D Problems

For three-dimensional problems we can form up different combinations of equations to arrive at a complete set of governing equations. For example:

		No. Equations	No. unknowns
Equilibrium	$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \ldots = 0$	3	6 stresses
Stress-strain	$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \ldots\right]$	6	6 strains
Compatibility	$2\frac{\partial^2 e_{yx}}{\partial y \partial z} - \frac{\partial^2 e_{yy}}{\partial z^2} + \ldots = 0$	3 (independent)	-
	TOTAL	12	12

Alternately, we could use the set

		No. Equations	No. unknowns
Equilibrium	$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \ldots = 0$	3	6 stresses
Stress-strain	$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_{xx} + \ldots\right]$	6	6 strains
Strain-displacement	$e_{xx} = \frac{\partial u_x}{\partial x} \quad \ldots$	6	3 displ.
	TOTAL	15	15

Of course, it is not necessary to deal with all these equations simultaneously. For example, we could place the strain-displacement relations into the stress-strain equations which would express the stresses directly in terms of the displacement gradients, and then place those stress-displacement gradient relations into the equilibrium equations to arrive at three equations for the three displacements only. Those three displacement equations are called Navier's equations.

Plane Stress Problems

For plane stress or plane strain problems, we can also work with different combinations. For example, for plane stress cases:

	No. Equations	No. Unknowns
Equilibrium		
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$$ $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$	2	3 stresses
Stress-strain		
$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(e_{xx} + \nu e_{yy}\right)$$ $$\sigma_{yy} = \frac{E}{1-\nu^2}\left(e_{yy} + \nu e_{xx}\right)$$ $$\sigma_{xy} = G\gamma_{xy}$$	3	3 strains
Compatibility		
$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$	1	-
TOTAL	6	6

Or, alternatively we can introduce the displacements to have

	No. Equations	No. Unknowns
Equilibrium		
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$$ $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$	2	3 stresses
Stress-strain		
$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(e_{xx} + \nu e_{yy}\right)$$ $$\sigma_{yy} = \frac{E}{1-\nu^2}\left(e_{yy} + \nu e_{xx}\right)$$ $$\sigma_{xy} = G\gamma_{xy}$$	3	3 strains
Strain-displacement		
$$e_{xx} = \frac{\partial u_x}{\partial x}$$ $$e_{yy} = \frac{\partial u_y}{\partial y}$$ $$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$	3	2 displ.
TOTAL	8	8

Plane Strain Problems

The plane strain equations are very similar to the plane stress case:

	No. Equations	No. Unknowns
Equilibrium		
$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$ $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$	2	3 stresses
Stress-strain		
$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(e_{xx} + \frac{\nu}{(1-\nu)} e_{yy}\right)$ $\sigma_{yy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(e_{yy} + \frac{\nu}{(1-\nu)} e_{xx}\right)$ $\sigma_{xy} = G\gamma_{xy}$	3	3 strains
Compatibility		
$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$	1	-
TOTAL	6	6

Or, alternatively we can introduce the displacements to have

	No. Equations	No. Unknowns
Equilibrium		
$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0$ $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$	2	3 stresses
Stress-strain		
$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(e_{xx} + \frac{\nu}{(1-\nu)} e_{yy}\right)$ $\sigma_{yy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(e_{yy} + \frac{\nu}{(1-\nu)} e_{xx}\right)$ $\sigma_{xy} = G\gamma_{xy}$	3	3 strains
Strain-displacement		
$e_{xx} = \frac{\partial u_x}{\partial x}$ $e_{yy} = \frac{\partial u_y}{\partial y}$ $\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$	3	2 displ.
TOTAL	8	8

Comparing the plane stress and plane strain cases, we see that except for the elastic constants these two cases are identical as far as the in-plane stresses and deformations. Of course the out-of-plane conditions are different since for the plane stress case we have $\sigma_{zz} = 0$ but the normal strain is given by

$$e_{zz} = \frac{-\nu}{(1-\nu)}(e_{xx} + e_{yy})$$

while for the plane strain case $e_{zz} = 0$ but the normal stress σ_{zz} is

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

But these out-of-plane stresses or strains are obtained from the behavior of the in-plane quantities so this difference does not affect how we solve either a plane stress or plane strain problem.

By replacing the elastic constants appropriately, we can turn the solutions for plane stress problems into plane strain problems and vice versa. For example, if we have a plane strain solution valid for (E, ν, G) and set

$$G' = G$$

$$\nu' = \frac{\nu}{(1-\nu)}$$

$$\frac{E'}{\left[1-(\nu')^2\right]} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \Rightarrow E' = \frac{E}{(1-\nu^2)}$$

we will have a corresponding plane stress solution valid for (E', ν', G'). Similarly, if we have a plane stress solution valid for (E, ν, G) and set

$$G' = G$$

$$\frac{\nu'}{(1-\nu')} = \nu \Rightarrow \nu' = \frac{\nu}{(1+\nu)}$$

$$\frac{E'(1-\nu')}{(1+\nu')(1-2\nu')} = \frac{E}{(1-\nu^2)} \Rightarrow E' = \frac{E(1+2\nu)}{(1+\nu)^2}$$

we will obtain a plane strain solution valid for (E', ν', G'). The table listed below summarizes these relationships

Given a plane stress solution for (E', ν', G')	substituting $E' = \frac{E}{1-\nu^2}$ $\nu' = \frac{\nu}{(1-\nu)}$ $G' = G$	will give a plane strain solution for (E, ν, G)
Given a plane strain solution for (E', ν', G')	substituting $E' = \frac{E(1+2\nu)}{(1+\nu)^2}$ $\nu' = \frac{\nu}{(1+\nu)}$ $G' = G$	will give a plane stress solution for (E, ν, G)