Chapter 5
Linear Elasticity

1 Introduction

The simplest mechanical test consists of placing a standardized specimen with its ends in the grips of a tensile testing machine and then applying load under controlled conditions. Uniaxial loading conditions are thus approximately obtained. A force balance on a small element of the specimen yields the longitudinal (true) stress as

$$\sigma = \frac{F}{A}$$

where F is the applied force and A is the (instantaneous) cross sectional area of the specimen. Alternatively, if the initial cross sectional area A_0 is used, one obtains the engineering stress

$$\sigma_e = \frac{F}{A_0}$$

For loading in the elastic regime, for most engineering materials $\sigma_e \approx \sigma$.

Likewise, the true strain is defined as

$$\epsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln(\frac{l}{l_0})$$

while the engineering strain is given by

$$\epsilon_e = \int_{l_0}^{l} \frac{dl}{l_0} = \frac{l - l_0}{l_0}$$

Again, for loading in the elastic regime, for most engineering materials $\epsilon_e \approx \epsilon$.

Linear elastic behavior in the tension test is well described by Hooke's law, namely

$$\sigma = E\epsilon$$

where E is the modulus of elasticity or Young's modulus. For most materials, this is a large number of the order of 10^{11} Pa.

Values of E can be readily determined by measuring the speed of propagation of longitudinal elastic waves in the material. Ultrasonic waves are induced by a piezoelectric device on the surface on the specimen and their rate of propagation accurately measured. The velocity of the longitudinal wave is given by

$$v_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}$$

Transverse wave propagation rates are also easily measured by ultrasonic techniques and the corresponding relationship is

$$v_T = \sqrt{\frac{G}{\rho}}$$

where G is the modulus of elasticity in shear or shear modulus.

The shear modulus is involved in the description of linear elastic behavior under shear loading, as encountered, for instance during torsion testing of thin walled pipes. In this case, if the shear stress is τ and the shear strain γ, Hooke's law is

$$\tau = G\gamma$$

2 Generalized Hooke's Law

The statement that the component of stress at a given point inside a linear elastic medium are linear homogeneous functions of the strain components at the point is known as the generalized Hooke's law. Mathematically, this implies that

$$\sigma^{ij} = D^{ijkl}\epsilon_{kl}$$

where σ^{ij} and ϵ_{ij} are, respectively the stress and strain tensor components. The quantity D^{ijkl} is the tensor of elastic constants and it characterizes the elastic properties of the medium. Since the stress tensor is symmetric, the elastic constants tensor consists of 36 components.

The elastic strain energy W is defined as the symmetric quadratic form

$$W = \frac{1}{2}\sigma^{ij}\epsilon_{kl} = \frac{1}{2}D^{ijkl}\epsilon_{ij}\epsilon_{kl}$$

and has the property that $\sigma^{ij} = \partial W/\partial\epsilon_{ij}$. Because of the symmetry of W, the actual number of elastic constants in the most general case is 21. This number is further reduced in special cases that are of much interest in applications. For instance, for isotropic materials (elastic properties the same in all directions) the number of elastic constants is 2. For orthotropic materials (characterized by three mutually perpendicular planes of symmetry) the number of constants is 9. If the material exhibits symmetry with respect to only one plane, the number of constants is 13.

3 Stress-Strain Relations for Isotropic Elastic Solids

The generalized Hooke's law for isotropic solids is

$$\sigma_{\alpha\alpha} = 3K\epsilon_{\alpha\alpha}$$
$$\sigma'_{ij} = 2G\epsilon'_{ij}$$

where K and G are the elastic constants bulk modulus and shear modulus, respectively and the primes denote the stress and strain deviators.

Combination of the above with the definition of stress and strain deviation tensors yields the following commonly used forms of Hooke's law; for stress, in terms of strain

$$\sigma_{ij} = \lambda\epsilon_{\alpha\alpha}\delta_{ij} + 2G\epsilon_{ij}$$

and for strain, in terms of stress

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\alpha\alpha}\delta_{ij}$$

The constants λ and G are called Lame's constants, while E is Young's modulus and ν is Poisson's ratio. Any of the above elastic constants can be expressed in terms of the others and only two are independent. Values of the above elastic constants for a wide variety of engineering materials are readily available in handbooks.

For an isotropic elastic solid in a rectangular Cartesian system of coordinates, the constitutive equations of behavior then become

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]$$
$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$
$$\epsilon_{zx} = \frac{1}{2G}\sigma_{zx}$$

4 Stress-Strain Relations for Anisotropic Elastic Solids

It is conventional in studying elastic deformation of anisotropic materials to relabel the six stress and strain components as follows:

$$\sigma_{11} = \sigma_1$$

$$\sigma_{22} = \sigma_2$$
$$\sigma_{33} = \sigma_3$$
$$\sigma_{23} = \sigma_4$$
$$\sigma_{13} = \sigma_5$$
$$\sigma_{12} = \sigma_6$$

$$\epsilon_{11} = \epsilon_1$$
$$\epsilon_{22} = \epsilon_2$$
$$\epsilon_{33} = \epsilon_3$$
$$\epsilon_{23} = \frac{1}{2}\epsilon_4$$
$$\epsilon_{13} = \frac{1}{2}\epsilon_5$$
$$\epsilon_{12} = \frac{1}{2}\epsilon_6$$

With the new notation and using the summation convention, Hooke's law becomes

$$\sigma_i = C_{ij}\epsilon_j$$

or equivalently

$$\epsilon_i = S_{ij}\sigma_j$$

where C_{ij} and S_{ij} are, respectively the elastic stiffness and compliance matrices. Depending on the symmetries existing in the material, only a few components of the above matrices are nonzero. For instance, for single crystals with cubic structure only C_{11}, C_{12} and C_{44} are nonzero. Values of the components of the above matrices for a variety of anisotropic materials are readily available in handbooks.

5 Formulation of Linear Elastic Problems

For steady state conditions, the governing equations of the isotropic linear elastic solid are, the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \sigma_{ij,j} + X_i = 0$$

the stress-strain relations

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij}$$

and the small displacement, strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

Admissible strain fields are those that satisfy the equations of compatibility. The three equilibrium equations together with the six stress-strain relations constitute a set of nine equations for the nine unknowns u_i, σ_{ij}. One can show this system is complete, yields an unique solution under suitable boundary conditions and the resulting strain satisfies the compatibility relations.

The above system of equations must be solved in each particular case subject to appropriate boundary conditions. There are two fundamental boundary value problems of elasticity:

- Problem 1: Determine the stress and strain fields inside the elastic body subject to specified values of the displacement u_i at its boundary.
- Problem 2: Determine the stress and strain fields inside the elastic body subject to a specified values of the surface tractions T_i at its boundary.

Because of the nature of the boundary conditions associated with the two fundamental problems above, it is convenient to produce formulations of the elasticity problem involving only displacements or stresses.

Specifically., combining the strain-displacement relationships with Hooke's law and subsequent introduction of the result into the equilibrium equations yields the Navier equations

$$Gu_{i,jj} + (\lambda + G)u_{j,ji} + X_i = 0$$

The above is then a set of three equations for the three unknowns u_i.

Alternatively, combination of the compatibility equations, Hooke's law and the equilibrium equations yields the Beltrami-Michell equations

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}X_{k,k}\delta_{ij} - (X_{i,j} + X_{j,i})$$

An interesting special case of the equilibrium equations is obtained if the body force vanishes. In this case, one can readily show that the two invariants, the dilatation $e = \epsilon_{\alpha\alpha}$ satisfies

$$\nabla^2 e = 0$$

and the mean stress $\sigma = \frac{1}{3}\sigma_{\alpha\alpha}$ satisfies

$$\nabla^2\sigma = 0$$

Furthermore

$$\nabla^4 u_i = 0$$

$$\nabla^4 \epsilon_{ij} = 0$$

$$\nabla^4 \sigma_{ij} = 0$$

i.e. the dilatation and the mean stress are harmonic functions while the displacement vector components, the strain tensor components and the stress tensor components are all biharmonic functions.

6 Strain Energy Function

An important quantity in continuum mechanics is the energy associated with deformation. The specific strain energy of the linear elastic material W was defined above as

$$W = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$$

It has the properties

$$\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij}$$

and

$$\frac{\partial W}{\partial \sigma_{ij}} = \epsilon_{ij}$$

It can be shown that, as long as the strain energy function exists and is positive definite, the two fundamental boundary value problems of elasticity have unique solutions. Non-unique solutions may occur if W is not positive definite (e.g. buckling, inelastic deformation) or when the linear equations break down (e.g. finite deformation, forces with memory).

7 Torsion of Elastic Bars

Saint-Venant produced solutions for the torsion problem of a long bar (aligned with the $z-$ direction), under small twist conditions by assuming the vector components of displacement to be given by

$$\begin{aligned} u &= -\alpha z y \\ v &= \alpha z x \\ w &= \alpha \phi(x, y) \end{aligned}$$

where α is the angle of twist per unit length of the bar and $\phi(x, y)$ is the warping function.

It can be shown that the warping function satisfies Laplace's equation

$$\nabla^2 \phi = 0$$

subject to the condition

$$\frac{\partial \phi}{\partial n} = y \cos(x, n) - x \cos(y, n)$$

at the lateral boundary of the bar and is therefore an harmonic function.

Alternatively, Prandtl proposed, in analogy with the stream function of hydrodynamics, the use of the stress function $\psi(x, y)$ defined by

$$\sigma_{xz} = \frac{\partial \psi}{\partial y}$$
$$\sigma_{yz} = -\frac{\partial \psi}{\partial x}$$

It can be shown that the equations of elasticity for the torsion problem in terms of the stream function are equivalent to the problem of solving

$$\nabla^2 \psi = -2G\alpha$$

subject to

$$\psi = 0$$

on the boundary of the bar.

For instance for the bar of elliptical cross section (axes a and b), a suitable expression for the stress function is

$$\psi = k[(\frac{x}{a})^2 + (\frac{y}{b})^2) - 1]$$

With the above one can readily obtain closed form expressions for the shearing stresses τ_{zx}, τ_{zy} and $\tau_{z\alpha} = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$, the torque $M_t = 2 \int \int \psi dx dy$, angle of twist α and the warpage w.

8 Bending of Beams

Consider a cantilever beam (length L, moment of inertia of the cross section I), loaded at one end with transverse force P. Select a rectangular Cartesian system of coordinates with the z-axis aligned with the axis of the beam and the $x-y$ plane coincident with its cross-section. The equilibrium equations are

$$\frac{\partial \sigma_{zx}}{\partial z} = 0$$
$$\frac{\partial \sigma_{yz}}{\partial z} = 0$$

and

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{Px}{I} = 0$$

It can be shown that in order to satisfy the compatibility requirement (expressed in terms of the stress tensor - Beltrami-Michell equations), the above problem can be represented by the following equivalent one in terms of the stress function ψ, namely, determine the function $\psi(x, y)$ satisfying

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = \frac{\nu}{1+\nu} \frac{Py}{I} - \frac{df}{dy}$$

subject to the condition $\psi = 0$ on the boundary of the cross section of the beam, where $f(y)$ is a conveniently defined but arbitrary function introduced by Timoshenko.

Specifically, for the beam of circular cross section (radius r), a suitable form for $f(y)$ is

$$f(y) = \frac{P}{2I}(r^2 - y^2)$$

with this, the above equation for ψ becomes

$$\nabla^2 \psi = \frac{1+2\nu}{1+\nu} \frac{Py}{I}$$

with the solution

$$\psi = -\frac{1+2\nu}{8(1+\nu)} \frac{P}{I}(x^2 + y^2 - r^2)y$$

From this, the stress components are found to be

$$\sigma_{xz} = \frac{\partial \psi}{\partial y} - \frac{Px^2}{2I} + f(y) = \frac{(3+2\nu)}{8(1+\nu)} \frac{P}{I}(r^2 - x^2 - \frac{1-2\nu}{3+2\nu} y^2)$$

$$\sigma_{yz} = -\frac{\partial \psi}{\partial x} = -\frac{(1+2\nu)}{4(1+\nu)} \frac{Pxy}{I}$$

and, from elementary beam theory,

$$\sigma_{zz} = -\frac{P(L-z)x}{I}$$

9 Potential Methods

Helmholtz' theorem states that any reasonable vector field $\mathbf{u}$ can be expressed in general in terms of potentials, i.e.

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

where, here ϕ is the scalar potential and ψ is the vector potential. The scalar potential is directly related to the dilation e by

$$e = \phi, ii$$

while the vector potential relates to the rotation vector ω_i by

$$-2\omega_i = \psi_i, jj$$

A simple example of the use of potentials in elasticity is Lame's strain potential function ϕ defined by

$$2Gu_i = \frac{\partial\phi}{\partial x_i}$$

It can be shown that in the absence of body forces the potential is an harmonic function, i.e.

$$\nabla^2\phi = 0$$

and the stress tensor components are given by

$$\sigma_{ij} = \phi_{,ij}$$

Since harmonic functions have long been studied and are well known, they can be readily used to obtain solutions to many practical problems. For instance, for a hollow sphere (inner radius a, outer radius b) subjected to inner pressure p and outer pressure q, one can use the potential function

$$\phi = \frac{C}{R} + DR^2$$

where C and D are constants and $R^2 = x^2 + y^2 + z^2$ to determine the stress at $(R, 0, 0)$. The values of the constants C and D are determined by incorporating the boundary conditions to yield

$$\sigma_{RR} = -p\frac{(b/R)^3 - 1}{(b/a)^3 - 1} - q\frac{1 - (a/R)^3}{1 - (a/b)^3}$$

and

$$\sigma_{\theta\theta} = \frac{p}{2}(\frac{(b/R)^3 + 2}{(b/a)^3 - 1}) - \frac{q}{2}(\frac{(a/R)^3 + 2}{1 - (a/b)^3})$$

Many other interesting problems can be solved using the method of potentials.

10 Two-Dimensional Problems in Rectangular Cartesian Coordinates

In Cartesian coordinates (x, y), the Airy stress function $\Phi(x, y)$ is implicitly defined by the equations

$$\frac{\partial^2 \Phi}{\partial y^2} = \sigma_{xx} - V$$
$$\frac{\partial^2 \Phi}{\partial x \partial y} = -\sigma_{xy}$$
$$\frac{\partial^2 \Phi}{\partial x^2} = \sigma_{yy} - V$$

where the potential V is implicitly given by

$$\nabla V = -X$$

where X is the body force vector.

If $X = 0$ then

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^4} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Hence, the stress function is a biharmonic function.

11 Two-Dimensional Problems in Polar Coordinates

The equations of equilibrium in two-dimensional polar coordinates (r, θ), where the body force acts only along the $r-$direction are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

One can show that the appropriate form of the stress function equation required to satisfy compatibility is

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}) = 0$$

where

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r}$$

and

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

An important special case is obtained when the stress distribution is symmetrical about an axis. The stress function equation in this case becomes

$$\frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0$$

One can show that a fairly general solution of the above is of the form

$$\Phi = A \log r + Br^2 \log r + Cr^2 + D$$

11.1 Pressurized Cylindrical Tube

Consider a long cylindrical tube subjected to uniform pressures at its inner and outer surfaces ($-p$ at $r = a$ and $-q$ at $r = b$). Axial symmetry can be assumed and body forces may be neglected. The equilibrium equation becomes

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

Moreover, Hooke's law reduce to

$$\epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = \frac{du}{dr}$$

and

$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) = \frac{u}{r}$$

where u is the radial displacement.

It can be shown that an appropriate form for the stress function that yields physically meaningful, single-valued displacement functions in this case is

$$\Phi = A \log r + Cr^2 + D$$

Taking derivatives of this and substituting the stated boundary conditions yields expressions for the unknown constants A and C and the stress field is given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} = -p\frac{(b/r)^2 - 1}{(b/a)^2 - 1} - q\frac{1 - (a/r)^2}{1 - (a/b)^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} = p\frac{(b/r)^2 + 1}{(b/a)^2 - 1} - q\frac{1 + (a/r)^2}{1 - (a/b)^2}$$

And the displacement is given by

$$u = \frac{1-\nu}{E}\frac{(a^2p - b^2q)r}{b^2 - a^2} + \frac{1+\nu}{E}\frac{(p-q)a^2b^2}{(b^2-a^2)r}$$

In this particular case, the problem can also be solved without involving the stress function by solving the equidimensional differential equation that results from substituting Hooke's law and the strain-displacement relationships into the equilibrium equation.

11.2 Bending of a Curved Bar

For the bending of a curved bar (inner curvature radius a, outer curvature radius b), by a force P acting towards the origin at one end while the other end is clamped, one has

$$\Phi = (d_1 r^3 + \frac{c_1'}{r} + d_1' r \log r)\sin\theta$$

where

$$\begin{aligned} d_1 &= \frac{P}{2N} \\ c_1' &= -\frac{Pa^2b^2}{2N} \\ d_1' &= -\frac{P}{N}(a^2 + b^2) \end{aligned}$$

where $N = a^2 - b^2 + (a^2 + b^2)\log(b/a)$.

11.3 Rotating Disks

Consider the problem of a solid disk of material with density ρ, radius b, of uniform thickness and rotating about its center with angular velocity ω. If the disk thickness is small compared to its radius, both, radial and tangential stresses can be regarded approximately constant though the thickness Moreover, because of the symmetry, the stress components can be regarded as functions of r only and the shear stresses vanish. The equilibrium equations reduce to a single one, namely

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = \frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

where $X_r = \rho\omega^2 r$.

It can be shown that a stress function F defined by the relationships

$$F = r\sigma_r$$

and

$$\frac{dF}{dr} = \sigma_\theta - \rho\omega^2 r^2$$

produces stresses satisfying the equilibrium equation.

The strain tensor components in this case are

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

where u is the radial displacement and Hooke's law has been used.

By eliminating u from the above equations and using Hooke's law, the following differential equation for F is obtained

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + (3+\nu)\rho\omega^2 r^3 = 0$$

The general solution is of the form

$$F = Cr + \frac{C_1}{r} - \frac{3+\nu}{8}\rho\omega^2 r^3$$

Particular solutions to specific problems can now be obtained by introducing boundary conditions. For instance, for a solid disk, the stresses are finite at the center and σ_r vanishes at its outer radius $r = b$. The resulting stresses are then

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 b^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

Alternatively, for a disk with a hole at the center (radius a), since $\sigma_r = 0$ at both $r = a$ and $r = b$, the stresses become

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2)$$

Note that as the size of the hole approaches zero, the maximum tangential stress does not converge to the value obtained for the solid disk. This is the effect of stress concentration associated with the presence of the hole.

11.4 Plate with a Hole under Uniaxial Tension

Consider a large plate with a circular hole (radius a) in the middle and under uniform uniaxial tension S along the $x-$direction.

Far away from the hole (for radii $b >> a$), the stresses are given by

$$(\sigma_r)|_{r=b} = \frac{S}{2}(1 + \cos(2\theta))$$

and

$$\tau_{r\theta} = -\frac{S}{2}\sin(2\theta)$$

where θ is the angle between the position vector and the positive $x-$axis measured clockwise.

It can be shown that the above stresses can be obtained from the stress function

$$\phi = f(r)\cos(2\theta)$$

by using the equations

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

and

$$\tau_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \phi}{\partial \theta})$$

By requiring the stress function to satisfy

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}) = 0$$

the compatibility condition is automatically satisfied. Introducing the assumed relationship for ϕ yields

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2})(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}) = 0$$

This can be readily solved yielding the following general solution for the stress function

$$\phi = (AR^2 + Br^4 + \frac{C}{r^2} + D)\cos(2\theta)$$

Finally, introducing the far-field stress conditions (at $r = b$) and the condition of zero external forces at the hole boundary yields

$$\sigma_r = \frac{S}{2}(1 - (\frac{a}{r})^2) + \frac{S}{2}(1 + 3(\frac{a}{r})^4 - 4(\frac{a}{r})^2)\cos(2\theta)$$

$$\sigma_\theta = \frac{S}{2}(1 + (\frac{a}{r})^2) - \frac{S}{2}(1 + 3(\frac{a}{r})^4)\cos(2\theta)$$

$$\tau_{r\theta} = -\frac{S}{2}(1 - 3(\frac{a}{r})^4 + 2(\frac{a}{r})^2)\sin(2\theta)$$

12 Thermodynamics of Elastic Bodies

Thermodynamics accounts for the effects of thermal phenomena on the response of materials to mechanical loads. The first law of thermodynamics states that energy is conserved. The second law states that entropy is always created. The combined statement of the first and second laws of thermodynamics as applied to solid bodies is

$$dU = TdS + \frac{1}{\rho}\sigma_{ij}d\epsilon_{ij}$$

where U is the specific internal energy or energy per unit mass and S is the specific entropy or entropy per unit mass.

The Helmholtz and Gibbs free energy functions are often useful in solving practical problems. They are defined respectively as

$$F = U - TS$$

and

$$G = U - TS - \frac{1}{\rho}\sigma_{ij}\epsilon_{ij}$$

The equations of thermodynamics can be used to obtain expressions of the condition required for thermodynamic equilibrium. Two equivalent statements of the condition are

- A system is in thermodynamic equilibrium if for all possible variations of the state taking place at constant energy, the entropy change is negative, i.e. $(\Delta S)|_U < 0$, the entropy is maximal.
- A system is in thermodynamic equilibrium if for all possible variations of the state taking place at constant entropy, the energy change is non-negative, i.e. $(\Delta U)|_S \geq 0$, the energy is minimal (or positive definite).

The laws of thermodynamics impose definite restrictions on the mathematical form of constitutive equations. In the case of linear elastic bodies, combination of Hooke's law with the definition of strain energy yields

$$dW = \sigma_{ij} d\epsilon_{ij} = \lambda e de + 2G\epsilon_{ij} d\epsilon_{ij}$$

For a point in a body going from zero stress to a particular stressed state, the total strain energy becomes

$$W = \int dW = \frac{\lambda}{2} e^2 + G\epsilon_{ij}\epsilon_{ij}$$

or, in terms of the second invariant of the strain deviation tensor, $J_2 = \frac{1}{2}\epsilon'_{ij}\epsilon'_{ij}$,

$$W = \frac{1}{2} K e^2 + 2GJ_2$$

Since, thermodynamics requires the energy to be positive definite, necessarily

$$E > 0$$

and

$$-1 < \nu < \frac{1}{2}$$

The strain energy function must be modified when the body is subjected to thermal loads in addition to mechanical ones since there is thermal expansion. The appropriate form of the strain energy in this case is

$$W = -\beta_{ij}(T - T_0)\epsilon_{ij} + \frac{1}{2} C'_{ijkl}\epsilon_{ij}\epsilon_{kl}$$

where β_{ij} is a tensor of thermal expansion coefficients and T_0 is a reference temperature. With this, the stress becomes

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(T - T_0)$$

As in the isothermal case, the number of elastic constants is significantly reduced for an isotropic body and the expressions for stress, strain and energy become

$$\sigma_{ij} = \lambda e \delta_{ij} + 2G\epsilon_{ij} - \frac{E\alpha}{1 - 2\nu}(T - T_0)\delta_{ij}$$

$$\epsilon_{ij} = -\frac{\nu}{E} s\delta_{ij} + \frac{1 + \nu}{E}\sigma_{ij} + \alpha(T - T_0)\delta_{ij}$$

and

$$W = G[\epsilon_{ij}\epsilon_{ij} + \frac{\nu}{1-2\nu}e^2 - \frac{2(1+\nu)}{1-2\nu}\alpha(T-T_0)e]$$

where α is the coefficient of thermal expansion, $e = \epsilon_{ii}$ and $s = \sigma_{ii}$.

With the above, expressions for the various thermodynamic functions applicable to isotropic linear elastic solid can be derived. For instance, the internal energy is given by

$$U = \frac{1}{\rho}[\frac{1}{2}Ke^2 + 2GJ_2 + \frac{E\alpha}{1-2\nu}T_0 e + \rho(C_v)|_{e_{ij}=0}(T-T_0)]$$

and for the Gibbs free energy

$$G = \frac{1}{\rho}[\frac{\nu}{2E}\sigma^2 - \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \alpha(T-T_0)\sigma + \rho\int_{T_0}^{T} dT \int_{T_0}^{T}(C_p)|_{\sigma_{ij}=0}\frac{dT'}{T'}]$$

Moreover, various equations connecting the thermal and mechanical properties of the elastic solid are also readily derived. One example is the formula

$$C_p - C_v = \frac{T}{\rho}\frac{\partial \epsilon_{ij}}{\partial T}\frac{\partial \sigma_{ij}}{\partial T}$$

13 The Finite Element Method

13.1 Introduction

The finite element method is the most widely used numerical technique for the solution of complex problems in solid mechanics. The method is based on first subdividing the domain of interest into a collection of small volumes called finite elements which are connected at nodal locations. The solution is approximated inside the elements from the values at the nodes using special functions called shape functions and the result is introduced into the variational (virtual power) formulation of the problem. The result is a set of algebraic equations that upon solution yield the values of the desired unknown at the nodal locations.

13.2 Variational Methods

Recall that the boundary value problem consisting of finding the function $u(x)$ such that

$$-\frac{d^2u}{dx^2} = -u'' = f$$

subject to the boundary conditions $u(0) = u(1) = 0$ is equivalent to the variational problem consisting of determining the function $u(x)$ such that

$$a(u, v) = (f, v)$$

where

$$a(u,v) = \int_0^1 -u''dx = \int_0^1 u'v'dx$$

and

$$(f,v) = \int_0^1 fvdx$$

and also to the problem of finding the function $u(x)$ that minimizes the functional

$$I(u) = \frac{1}{2}a(u,v) - (f,v)$$

where v is a reasonably well behaved but rather arbitrary function.

When one looks for approximate solutions to the integrated forms of the problem by the Galerkin method or by the Ritz method, one seeks for solutions in a finite dimensional subspace of the actual solution space.

In the Ritz method, one searches for an approximation to the function that minimizes the functional. The method starts by introducing a set of basis functions $\{\phi_i(x)\}_{i=1}^N$ and expresses the approximate solution as the *trial function* $u_N \approx u$ as

$$u_N = \sum_{i=1}^N a_i\phi_i$$

The functional obtained using this approximation then becomes

$$I(u_N) = \frac{1}{2}a(u_N,v) - (f,v)$$

Now, the function v being arbitrary, is selected as the *test function*

$$v = v_N = u_N = \sum_{i=1}^N a_i\phi_i$$

and is substituted into the functional expression to give

$$I(u_N) = \frac{1}{2}a(\sum_{i=1}^N a_i\phi_i, \sum_{i=1}^N a_i\phi_i) - (f, \sum_{i=1}^N a_i\phi_i)$$

where

$$a(\sum_{i=1}^N a_i\phi_i, \sum_{i=1}^N a_i\phi_i) = \int_0^1 (a_i \sum_{i=1}^N \frac{d\phi_i}{dx})^2 dx$$

and

$$(f, \sum_{i=1}^{N} a_i \phi_i) = \int_0^1 f(\sum_{i=1}^{N} a_i \phi_i) dx$$

Finally, the unknown coefficients $a_i, i = 1, 2, ..., N$ are determined by solving the system of algebraic equations obtained from the extremum conditions. i.e.

$$\frac{\partial I}{\partial a_1} = \frac{\partial I}{\partial a_2} = ... = \frac{\partial I}{\partial a_N} = 0$$

which in matrix notation are simply written as

$$\mathbf{Ka} = \mathbf{F}$$

In the Galerkin method, one seeks directly for an approximate solution of $a(u, v) = (f, v)$. Like in the Ritz method, in the Galerkin approach one starts by introducing a set of basis functions $\{\phi_j(x)\}_{j=1}^N$ satisfying the stated boundary conditions and expresses the approximate solution as the *trial function* $u_N \approx u$ as

$$u_N = \sum_{j=1}^{N} a_j \phi_j$$

However, the *test functions* are selected so as to be identical to the basis functions, i.e. $\{v_i\}_{i=1}^N = \{\phi_i(x)\}_{i=1}^N$.

The resulting discrete form of the problem is then

$$\sum_{i=1}^{N} a_j a(\phi_j, \phi_i) = (f, \phi_i)$$

for all $i = 1, 2, ..., N$. Here

$$a(\phi_j, \phi_i) = \int_0^1 \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} dx$$

and

$$(f, \phi_i) = \int_0^1 f \phi_i dx$$

for $i = 1, 2, ..., N$

Using matrix notation, the above is simply written as

$$\mathbf{Ka} = \mathbf{F}$$

Solving the algebraic problem yields the values of the coefficients $a_i, i = 1, 2, ..., N$.

The system of equations obtained with the Galerkin method is identical to the one obtained using the Ritz method. However, the Galerkin method is of more general applicability since it works even when the problem of minimizing the functional has no solution. The finite element method is obtained by implementing the Galerkin (or Ritz) procedure with a very particular choice of basis functions.

13.3 The Finite Element Method

The finite element method is a numerical procedure designed to find approximate solutions of boundary value problems. The method is a direct implementation of the Galerkin (or Ritz) procedure in which the chosen basis functions are finite element basis functions (also called sometimes finite element shape functions). These functions are of very simple form (piecewise polynomials of low order are most common) and their most distinctive feature is that they are nonzero only in a small subregion of the computational domain, i.e. they have a local character. Finite element basis functions are then said to posses local or compact support.

This is in direct contrast with the classic Galerkin or Ritz approaches where basis functions valid over the entire computational domain are used. Another important feature of the finite element basis functions is that the basis function associated with a particular element partially overlap with those of immediately adjacent elements.

Improved approximation accuracy is readily obtained with the finite element method by simply increasing the number of subdomains (i.e. by decreasing their size) whitout increasing the complexity of the polynomials used to represent the solution inside each individual subregion.

By implementing the Galerkin process on individual subregions another powerful feature of the method emerges. Since the coordinate functions of adjacent subregions overlap partially with each other, the resulting system of algebraic equations can be constructed and assembled together by the computer just as it is created. This feature makes it a trivial process to determine increasingly better approximations by refinement of the initial subdivision. Moreover, the matrix associated with the resulting system has a sparse structure and efficient solution methods are applicable. This very desirable feature is absent in the classic implementation of the Galerkin or Ritz methods and constitutes an important disadvantage in those cases.

The simplest description of the fundamental characteristics of finite elements is obtained for one dimensional systems. Finite element characteristics for two and three dimensional elements are based on the same ideas but involve additional complexities of description that distract from the basic points. In this section basic characteristics of selected one dimensional elements are described. The implementation of the Ritz-Galerkin finite element method is also covered.

In sum, a finite element is a triple of entities (K, P_K, Σ) where

- K is a simple geometrical object, such as a line segment in one dimension, a triangle in two dimensions and a parallelepiped in three dimensions,
- P_K is a finite dimensional linear space of basis functions defined on K, and
- Σ is a set of degrees of freedom.

A finite element is then the totality of ingredients associated with the definition of the global approximating function inside the subdomain. Since the method produces approximations to

global functions from functions having only local support, it is key to be able to distinguish and to relate the global and local descriptions. Therefore, one needs to carefully specify both local and global parameters as well as their relationship.

13.4 Global and Local Coordinates and Basis Functions for Unidimensional Systems

Local parameters are specified at the element level. For instance, consider the one dimensional case. A simple subdivision of a given domain $x \in [a, b]$ is readily obtained by introducing a collection of N contiguous subdomains $[x_1 = a, x_2], [x_2, x_3], ..., [x_i, x_{i+1}], ..., [x_N, x_{N+1} = b]$. The points $x_1, x_2, x_3, ..., x_i, ..., x_N, x_{N+1}$ are called nodes and each contiguous pair of them, $[x_i, x_{i+1}]$, represents the boundaries of the element that spans the space h_i between the nodes. In this case, each element has two nodes one located at each end of the element. The location of all these points is given with reference to a global system of coordinates. If nodes are numbered based on their position is terms of global coordinates one obtains global node numbers.

In the local description of the element bounded by nodes $[x_i, x_{i+1}]$, the positions measured with respect to the global system of coordinates are transformed into coordinates $[\xi_1, \xi_2] = [-1, +1]$ referred to a system of coordinates with its origin in the center of the element, i.e. $[x_i, x_{i+1}] \rightarrow [\xi_1, \xi_2]$. The affine or linear transformation is simplest, i.e.

$$\xi(x) = \frac{2x - x_i - x_{i+1}}{h_i}$$

The inverse transformation is

$$x(\xi) = \frac{h_i \xi + x_i + x_{i+1}}{2}$$

If for any given element, node numbers are assigned based on the local coordinates one obtains local node numbers. For instance, the node with global number i is the same as local node 2 for element i and also as local node 1 for element $i + 1$. Since in finite element work on uses basis functions with local support and thus implements the Galerkin method at the element level to subsequently assemble the resulting equations, the distinction between local and global representations is quite important.

In the finite element method one starts by using simple interpolation functions as basis functions at the element level and then proceeds to represent the solution in the entire domain by collecting the contributions associated with each element. The basis functions involved at the element level are called local basis functions while the ones that produce the solution in the entire domain are called global basis functions. Clearly, the two sets of basis functions are closely related.

The role of the local basis functions is to generate the value of the approximated quantity inside the element from the values at the nodes by interpolation. Assuming again a one dimensional system and the simple subdivision introduced above, consider an arbitrary interior

element e_i where the index i denotes the global node number. If the values of the required solution at x_i and x_{i+1} are $u(x_i) = u_i$ and $u(x_{i+1}) = u_{i+1}$, respectively, the approximate values of $u(x)$, $u(x)^{e_i}$ inside the element are calculated by simple interpolation as follows

$$u^{e_i}(x) = u_i\phi_1^i + u_{i+1}\phi_2^i$$

Here, the local finite element basis functions for element e_i, ϕ_1^i and ϕ_2^i are defined as

$$\phi_1^i = \frac{x_{i+1} - x}{h_i}$$

$$\phi_2^i = \frac{x - x_i}{h_i}$$

where $h_i = x_{i+1} - x_i$ is the element size and all the positions are measured in the global coordinate system. Note that the indices 1 and 2 on the local basis functions refer to the local node numbers for the element.

For simplicity, assume all finite elements are of equal size (uniformly spaced mesh) so that $h_1 = h_2 = ... = h_i = h$. Now, the approximate solution on the entire domain $u_h(x)$ is represented as a linear combination of the finite element basis functions, i.e.

$$u_h(x) = u_1\phi_1 + u_2\phi_2 + ... + u_i\phi_i + u_N\phi_N = \sum_{i=1}^{N} u_i\phi_i$$

However, note that here, $\phi_i, i = 1, 2, ..., N$ are global finite element basis functions.

There is a simple relationship between local and global finite element basis functions, specifically, in element 1,

$$\phi_1 = \phi_1^1$$

in element 2

$$\phi_2 = \begin{cases} \phi_2^1 & \text{if } x \in e_1 \\ \phi_1^2 & \text{if } x \in e_2 \end{cases}$$

in element e_i,

$$\phi_i = \begin{cases} \phi_2^{i-1} & \text{if } x \in e_{i-1} \\ \phi_1^i & \text{if } x \in e_i \end{cases}$$

and in element e_N

$$\phi_N = \phi_2^N$$

Using the introduced notation the Galerkin finite element method equations can be expressed as

$$\sum_{i=1}^{N} u_j a(\phi_j, \phi_i) = (f, \phi_i)$$

for all $i = 1, 2, ..., N$. Here

$$a(\phi_j, \phi_i) = \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx$$

and

$$(f, \phi_i) = \int_0^1 f\phi_i dx$$

for $i = 1, 2, ..., N$.

Using matrix notation, the above is simply written as

$$\mathbf{Ku} = \mathbf{F}$$

Solving the algebraic problem yields the values of the nodal values $u_i, i = 1, 2, ..., N$. The matrix $\mathbf{K}$ is called the finite element stiffness matrix, the colum vector $\mathbf{F}$ is the force vector and the column vector $\mathbf{u}$ is the vector of unknown nodal values of u.

The same thing can be done to express the Ritz finite element method equations using the new notation. Specifically, the functional $I(u_h)$ is given by

$$I(u_h) = \frac{1}{2}a(\sum_{i=1}^{N} u_i\phi_i, \sum_{i=1}^{N} u_i\phi_i) - (f, \sum_{i=1}^{N} u_i\phi_i)$$

where

$$a(\sum_{i=1}^{N} u_i\phi_i, \sum_{i=1}^{N} u_i\phi_i) = \int_0^1 (\sum_{i=1}^{N} u_i\frac{d\phi_i}{dx})^2 dx$$

and

$$(f, \sum_{i=1}^{N} a_i\phi_i) = \int_0^1 f(\sum_{i=1}^{N} u_i\phi_i)dx$$

Finally, introduction of the extremum conditions

$$\frac{\partial I}{\partial u_1} = \frac{\partial I}{\partial u_2} = ... = \frac{\partial I}{\partial u_N}$$

yields the system

$$\mathbf{Ku} = \mathbf{F}$$

which is identical to the one obtained using Galerkin's method.