Basic concepts and models in continuum damage mechanics

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RÉSUMÉ. Dans cet article, on présente d'abord quelques éléments de base de la modélisation macroscopique de l'endommagement. On rappelle notamment le cadre thermodynamique des processus irréversibles et son application à la modélisation de l'endommagement isotrope. L'étude de l'anisotropie induite par l'endommagement est ensuite traitée en considérant une variable tensorielle d'ordre 2. Enfin, l'article s'achève sur une contribution originale dans laquelle on s'intéresse à la modélisation des effets unilatéraux liés à la refermeture des microfissures.

ABSTRACT. In this paper, we present some basic elements of macroscopic modelling of damage. We then recall the general approach of continuum damage based on the thermodynamics of irreversible processes and its application to isotropic damage modelling. The study of damage-induced anisotropy is treated by considering a second order tensorial damage variable. Finally, we present an original macroscopic approach through which is addressed the question of unilateral effects due to the microcracks closure.

MOTS-CLÉS : endommagement, poroélasticité, micromécanique, fissures.

KEYWORDS: damage, poroelasticity, micromechanics, cracks.

1. Some basic concepts in Continuum Damage Mechanics (CDM)

First of all, let us recall that the main objective of standard CDM is to propose a continuum-mechanics based framework allowing to characterize, represent and model at the macroscopic scale the effects of distributed defects and their growth on the material behavior. This requires the consideration of some basic concepts, some of which are recalled in the present section. To introduce these basic concepts of CDM[1], it proves useful to consider first an uniaxial tension experiment.

1.1. *The classical unixial damage theory*

Within the classical approach (see for instance (Lemaitre, 1996)), a very simple measure of the damage amplitude in a given plane is obtained by measuring the area of the intersection of all defects with that plane. For example, based on figure 1, it is readily observed that the effective area of the sample subjected to uniaxial tension is $S - S_D$. S_D represents the defects trace in the considered plane. The following positive scalar ω is then commonly considered as a damage variable in the above 1D experiment :
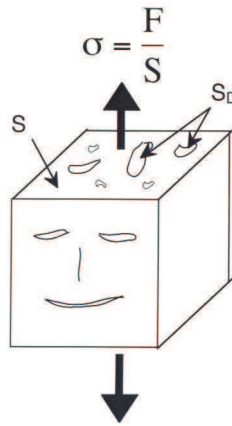
$$\omega = \frac{S_D}{S} \tag{1}$$

Figure 1. *Cross section of a damaged material*

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1. In the spirit of the Alert School, this section is inspired from a course given by M. Jirasek (Jirasek, 2002)

For the undamaged material, $S_D = 0$ and then $\omega = 0$. The damage being related to the growth of defects, ω may grow from 0 to a critical value often taken in litterature equal to 1 which corresponds to an entirely damaged material (effective area $S - S_D$ reduced to 0). Instead of the standard uniaxial stress $\sigma = \frac{F}{S}$, it is convenient to introduce for the damaged material the effective stress :

$$\tilde{\sigma} = \frac{F}{S - S_D} = \frac{F}{S(1 - S_D/S)} = \frac{\sigma}{1 - \omega} \qquad [2]$$

in which use has been made of [1].

Associated to a strain equivalence principle, the 1D effective stress $\tilde{\sigma}$ is related to the elastic strain of the material by the uniaxial Hooke's law :

$$\tilde{\sigma} = E\varepsilon \qquad [3]$$

where E is the elastic modulus of the undamaged material. It follows that the constitutive law for the standard stress σ takes the form :

$$\sigma = (1 - \omega)E\varepsilon \qquad [4]$$

For the uniaxial model formulation, equation [4] must be completed by the damage evolution law which can be considered in the form of a dependence between the damage variable ω and the applied load :

$$\omega = g(\varepsilon) \qquad [5]$$

A priori, the function g can be identified from uniaxial tension test. It must be noted that the relation between ω and ε is valid only in the monotonous loading regime. In an unloading and reloading phase, the damage variable kept its maximum value reached before. A classical way to describe in a unified manner these different loading regimes consists in introducing a variable κ which characterizes the maximum level of strain reached in the material before the current time $t : \kappa(t) = \max \varepsilon(\tau)$ for $\tau \leq t$. The damage evolution relation [5] can then be recast in the form :

$$\omega = g(\kappa) \qquad [6]$$

which remains valid for any kind of loading regime. The complete elastic response of the damaged material is schematized in figure 2. Instead of considering the function g, it is usual to introduce a limit state function :

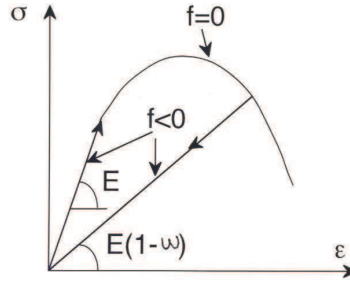$$f(\varepsilon, \kappa) = \varepsilon - \kappa \qquad [7]$$

Figure 2. *Elastic response of a damaged material during loading-unloading*

Equation [7] is completed by the classical Kuhn-Tucker condition :

$$f \leq 0; \quad \dot{\kappa} \geq 0; \quad \dot{\kappa}f = 0 \qquad [8]$$

The condition $f \leq 0$ indicates that ε can never be greater than κ, while the second condition means that κ cannot decrease. Besides, the second condition implies that κ can increase only if the current value of the strain is equal to κ.

In summary, the basic elements of the above uniaxial damage theory are as follows :

– the stress-strain law : $\sigma = E_s\varepsilon = (1 - \omega)E\varepsilon$; this relation appears as a classical Hooke's law with a secant modulus $E_s = (1 - \omega)E$ associated to the damaged material. A simple determination of the damage variable is then : $D = 1 - \frac{E_s}{E}$. It requires however a careful and accurate measure the elastic strain,

– a damage evolution law wich can be put in the form $\omega = g(\kappa)$ or a a limit state function f ; a first approach can consist to choose $f(\varepsilon, \kappa) = \varepsilon - \kappa$.

1.2. *A simple isotropic damage theory*

A straightforward 3D extension of the previous uniaxial theory is given by the well known Lemaitre-Chaboche (Lemaitre *et al.*, 1978) 3D elastic damage model. In this model, based on the various concepts presented in section 1, it is postulated that the stiffness $\mathbb{D}$ of the damaged material reads :

$$\mathbb{D} = (1 - \omega)\mathbb{D}_0 \qquad [9]$$

in which $\mathbb{D}_0$ denotes the elastic stiffness of the undamaged material and ω is defined by [1].

It is readily seen that the generalization of the uniaxial stress-strain constitutive law [4] takes the form :

$$\underline{\underline{\sigma}} = (1 - \omega)\mathbb{D}_0 : \underline{\underline{\varepsilon}} \qquad [10]$$

and the corresponding effective stress tensor $\underline{\underline{\tilde{\sigma}}}$ is given by :

$$\underline{\underline{\sigma}} = (1 - \omega)\underline{\underline{\tilde{\sigma}}} \qquad [11]$$

In the present three dimensional formulation of the isotropic damage model, only the Young modulus is affected, the Poisson ratio remains constant during the damage process. This is clearly a shortcoming of this simple model and will be corrected in section 2.

Let us come now to the 3D loading function f and to the damage evolution. f defines in the 3D strain space the domain of elasticity whose boundary corresponds to the strain states at which the damage will evolve. An immediate generalization of [7] reads :

$$f(\underline{\underline{\varepsilon}}, \kappa) = \varepsilon_{eq}(\underline{\underline{\varepsilon}}) - \kappa \qquad [12]$$

in which the equivalent strain ε_{eq} is a norm of $\underline{\underline{\varepsilon}}$ that needs to be choosen. A first simple choice ofor this can be :

$$\varepsilon_{eq} = \sqrt{\underline{\underline{\varepsilon}} : \underline{\underline{\varepsilon}}} \qquad [13]$$

Another choice can be the elastic energy (function of the strain) : $\varepsilon_{eq} = \sqrt{\underline{\underline{\varepsilon}} : \mathbb{D}_0 : \underline{\underline{\varepsilon}}}$.

1.3. *An isotropic damage modelling of concrete materials*

The two norms of $\underline{\underline{\varepsilon}}$ introduced before for the definitions of ε_{eq} leads to a symmetric elastic domain in tension and compression. However, several materials (rocks, concrete, ceramics) often show a dissymetric damage surface, the yield value in compression being several times the value in tension. In order to overcome these limitations, Mazars (Mazars, 1984) introduced two damage parameters, ω_t associated to a tension mechanism and ω_c devoted to the damage under compression. These two parameters, ω_t and ω_c, are evaluated from two evolution functions, g_t and g_c, which are assumed to depend both on a unique definition of the equivalent strain :

$$\varepsilon_{eq} = \sqrt{\sum_{I=1}^{3} <\varepsilon_I>^2} \qquad [14]$$

where $< . >$ is the Macauley bracket and ε_I are the principal strains. $< \varepsilon_I >$ denotes then the positive part of the principal value ε_I of the strain tensor.

The objective of the constitutive model initially proposed by (Mazars, 1984) is to capture the non linear response of geomaterials subjected to loading paths which involves extension. For a general loading path, different from uniaxial tension or compression, the value of damage which enters in the constitutive law is proposed as the following combination :

$$\omega = \alpha_t \omega_t + \alpha_c \omega_c \qquad [15]$$

where α_t and α_c are taken in the form (Pijaudier-Cabot *et al.*, 2001) :

$$\alpha_t = \sum_{I=1}^{3} \left[\frac{< \varepsilon_I^t >< \varepsilon_I >}{\varepsilon_{eq}^2} \right]^{\beta} ; \quad \alpha_c = \sum_{I=1}^{3} \left[\frac{< \varepsilon_I^c >< \varepsilon_I >}{\varepsilon_{eq}^2} \right]^{\beta} \qquad [16]$$

Obviously, in uniaxial tension $\alpha_t = 1$; $\alpha_c = 0$ and then $\omega = \omega_t$ while under uniaxial compression $\alpha_t = 0$; $\alpha_c = 1$ and $\omega = \omega_c$. An integrated form of the damage evolutuion is (see (Pijaudier-Cabot *et al.*, 2001)) :

$$\omega_t = g_t(\kappa) = \begin{cases} 0 & \text{if } \kappa \leq \kappa_0 \\ 1 - \frac{(1-A_t)\kappa_0}{\kappa} - A_t exp[-B_t(\kappa - \kappa_0)] & \text{if } \kappa \geq \kappa_0 \end{cases} \qquad [17]$$

and

$$\omega_c = g_c(\kappa) = \begin{cases} 0 & \text{if } \kappa \leq \kappa_0 \\ 1 - \frac{(1-A_c)\kappa_0}{\kappa} - A_c exp[-B_c(\kappa - \kappa_0)] & \text{if } \kappa \geq \kappa_0 \end{cases} \qquad [18]$$

where $\kappa_0 = \frac{f_t}{E}$ is the equivalent strain at the begining of the damage process. In this definition, f_t is the tensile strength of the material and E is the modulus of the undamaged material. Let us precise that the total number of the model parameters is 8 : the two elastic coefficients ; κ_0, A_t, B_t, A_c, B_c are materials parameters which can be derived from compressive and tensile tests ; the determination of the constant β requires a shear test (for simplicity β is often taken equal to 1).

An example of simulation of a concrete behavior by means of this model is reported in figure 3.

2. Thermodynamic framework of CDM : application to isotropic damage

The damage models presented in the preceeding section can be considered in a suitable framework given by the Thermodynamics of Irreversible Processes (TIP). In particular, this TIP framework allows a relatively easy modelling of anisotopic damage as in section 3.
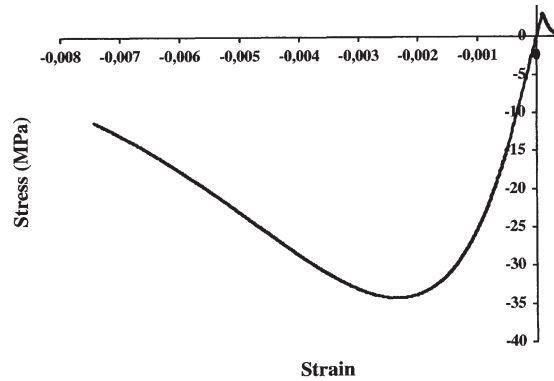
Figure 3. *Simulation of the uniaxial (tension and compression) response of a concrete. From (Pijaudier-Cabot and Mazars, 2001)*

2.1. *The internal damage variable approach*

Generally speaking, damage is defined as the modification of physical properties of materials in relation with the irreversible growth of microdefects. The macroscopic approach requires first the choice of some appropriate variables physically motivated and mathematically relevant for the description of the effects of microdefects at macroscale. Clearly enough, is desirable that the damage variable has some physical meaning and a mathematical coherence.

A representation of the damage can be obtained from a geometrical characterization of the cracks-like defects. When the damage process is due to the growth of spherical voids as in metallic materials, ω corresponds to the voids volume fraction, *i.e.* the porosity. In contrast, the non linear behavior of geomaterials may be related to microcracks growth. For a randomly oriented system of similar penny shaped microcracks for which anisotropic aspects can be disregarded, it was shown that the cracks density parameter ω (see (Budiansky *et al.*, 1976)) can be considered from a micromechanics point of view as a relevant damage variable (see also (Dormieux *et al.*, 2007) this issue) :

$$\omega = \mathcal{N}a^3 \tag{19}$$

where $\mathcal{N}$ denotes the crack density (number of cracks per unit volume) and a the radius of the penny-shaped cracks (see figure 4 for an example of a representive elementary volume and figure 5 for the geometrical description of a penny-shaped crack). For non circular microcracks, a generalization of [19] for the representation of the iso-

tropic damage has be given by (Kachanov, 1980) which includes the microcracks area and a shape parameter :

$$\omega = \mathcal{N} \sum_i \eta S_i^{3/2} \tag{20}$$

where S_i is the decohesion area related to an i-th cracks family and η is a dimensionless cracks-shape parameter.
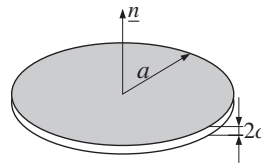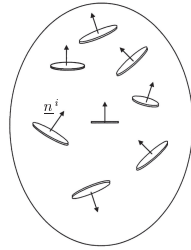

Figure 4. *R.e.v. of the cracked material*   **Figure 5.** *A Penny-shaped crack - $\epsilon = c/a$*

In the case of anisotropic damage, the cracks orientation $\underline{n}$ may be considered. A very simple representation of the anisotropy of the damage can be obtained by considering the following second order tensor $\underline{\underline{D}}$ based on the orientation tensor $\underline{n} \otimes \underline{n}$ (see again (Kachanov, 1980)) :

$$\underline{\underline{D}} = \sum_i \omega_i \underline{n}^i \otimes \underline{n}^i \tag{21}$$

in which ω_i and $\underline{n}^i$ are respectively the cracks density parameter and the orientation of a given cracks family i.

Such representation of damage allows to describe only materials symmetry up to orthotropy. The choice of the most appropriate tensorial representation for the damage variable corresponding to microcracking phenomena is still a much-debated question. The reader interrested by this topic can refer to the book (Krajcinovic, 1996).

2.2. *Thermodynamic potential and state laws*

The elastic damage behavior is characterized by the existence of a thermodynamic potential as function of the set of state variables. Strain formulation is considered (elastic strain $\underline{\underline{\varepsilon}}$ as observable variable) with isotropic damage (represented by a single damage variable ω). Assuming linear elasticity at constant damage, the free energy reads :

$$W(\underline{\underline{\varepsilon}}, \omega) = \frac{1}{2}\underline{\underline{\varepsilon}} : \mathbb{D}(\omega) : \underline{\underline{\varepsilon}} \tag{22}$$

where $\mathbb{D}(\omega)$ represents the isotropic stiffness tensor for a given damaged level, such that $\omega = 0$ corresponds to the virgin material. Due to the isotropy, the general form of $\mathbb{D}(\omega)$ is given later by [31]. The potential [22] must be continuously differentiable :

– with respect to the strain tensor in order to ensure the existence of the state laws which give : the stress tensor $\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}, \omega)$

$$\underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}, \omega) = \frac{\partial W}{\partial \underline{\underline{\varepsilon}}}(\underline{\underline{\varepsilon}}, \omega), \tag{23}$$

– with respect to the thermodynamic force $F^{\omega}(\underline{\underline{\varepsilon}}, \omega)$ associated to the damage variable :

$$F^{\omega}(\underline{\underline{\varepsilon}}, \omega) = -\frac{\partial W}{\partial \omega}(\underline{\underline{\varepsilon}}, \omega) = -\frac{1}{2}\underline{\underline{\varepsilon}} : \mathbb{D}'(\omega) : \underline{\underline{\varepsilon}} \tag{24}$$

where $\mathbb{D}' = \frac{\partial \mathbb{D}}{\partial \omega}$.

By similarity with the energy release rate in Linear Elastic Fracture Mechanics (see paper by C. Dascalu and by M. Jirasek, this issue), F^{ω} can be interpreted as a damage energy release rate.

2.3. *Damage surface and damage evolution law*

Before introducing the damage yield surface, let us analyze the dissipation $\mathcal{D}$ of the material in the presence of damage. One has :

$$\mathcal{D} = \underline{\underline{\sigma}} : \underline{\dot{\underline{\varepsilon}}} - \dot{W} = -\frac{\partial W}{\partial \omega}\dot{\omega} \tag{25}$$

Equation [25] suggests that $F^{\omega} = -\frac{\partial W}{\partial \omega}$ can be considered as the driving force of the damage process. Following (Marigo, 1985) and others, a quite classical form of the damage yield function f is the following one :

$$f(F^{\omega}, \omega) = F^{\omega} - \kappa(\omega) \tag{26}$$

κ is a scalar strictly positive function of ω which represents a resistance of the material to the damage propagation. Due to the dependence of κ on ω, the later plays the role of a hardening variable.

As an example, the following very simple form :

$$\kappa(\omega) = \kappa_0(1 + \eta\omega) \quad \text{with } \kappa_0 > 0 \text{ and } \eta > 0. \tag{27}$$

has been considered in several models of litterature.

For the damage evolution, it is usual to assume the normality rule ; the damage rate is then normal to the yield surface and reads then :

$$\begin{cases} \dot{\omega} = \dot{\Lambda}\dfrac{\partial f}{\partial F^{\omega}}(F^{\omega},\omega) = \dot{\Lambda} \\ \dot{\Lambda} = 0 \text{ if } f(F^{\omega},\omega) \leq 0, \ \dot{f}(F^{\omega},\omega) < 0 \\ \dot{\Lambda} > 0 \text{ if } f(F^{\omega},\omega) = 0, \ \dot{f}(F^{\omega},\omega) = 0 \end{cases}$$

[28]

in which the damage multiplier $\dot{\Lambda}$ is determined by the consistency condition. Reporting this evolution law in the rate form of the stress tensor, one gets the incremental formulation of the elastic damage law :

$$\dot{\underline{\underline{\sigma}}} = \mathbb{L} : \dot{\underline{\underline{\varepsilon}}}$$

[29]

with the tangent operator $\mathbb{L}$ given by the following expression :

$$\mathbb{L} = \begin{cases} \mathbb{D}(\omega) & \text{if } f(F^{\omega},\omega) \leq 0, \ \dot{f}(F^{\omega},\omega) < 0 \\ \mathbb{D}(\omega) - \dfrac{\dfrac{\partial F^{\omega}}{\partial \underline{\underline{\varepsilon}}} \otimes \dfrac{\partial F^{\omega}}{\partial \underline{\underline{\varepsilon}}}}{h} & \text{if } f(F^{\omega},\omega) = 0, \ \dot{f}(F^{\omega},\omega) = 0 \end{cases}$$

[30]

in [30], tensor $\mathbb{D}(\omega)$ is given by [49] and $h = \kappa_0\eta - \dfrac{\partial F^{\omega}}{\partial d}$.

2.4. *Different versions of the isotropic damage model*

In summary, the complete formulation of the isotropic model requires the choice of $\mathbb{D}(\omega)$ or equivalently the one of the thermodynamic potential. As the material remains isotropic, $\mathbb{D}(\omega)$ takes the following general form :

$$\mathbb{D}(\omega) = 3K(\omega)\mathbb{J} + 2\mu(\omega)\mathbb{K}.$$

[31]

where (K,μ) represent respectively the compressibility and shear moduli for the damaged material. $\mathbb{J}$ and $\mathbb{K}$ are the two isotropic fourth order projectors defined by $J = \frac{1}{3}\underline{\underline{1}} \otimes \underline{\underline{1}}$ and $\mathbb{K} = \mathbb{I} - \mathbb{J}$ with $\underline{\underline{1}}$ the unit second order tensor. $\mathbb{I}$ denotes the symmetric fourth order unit tensor.

Differents choices can be made for $\mathbb{D}(\omega)$. The model presented in subsection 1.2 corresponds to the simplest one and is defined by [9] which corresponds to the fact that damage affects the compressibility and shear modulus in the same maner. Consequently, Poisson ratio remains constant as already indicated.

A very efficient way to build macroscopic expression of $\mathbb{D}(\omega)$ is provided by micromechanics of microcracked materials. In practice, different schemes (dilute

scheme, Mori-Tanaka approach etc.)[2] can be considered. As an example, (Ponte-Castaneda *et al.*, 1995) established the following bound for randomly oriented penny-shaped microcracks with a spherical spatial distribution :

$$K(\omega) = K_0 \left(1 - \frac{48(1 - \nu_0^2)\omega}{27(1 - 2\nu_0) + 16\omega(1 + \nu_0)^2} \right) \qquad [32]$$

$$\mu(\omega) = \mu_0 \left(1 - \frac{480(1 - \nu_0)(5 - \nu_0)\omega}{675(2 - \nu_0) + 64\omega(4 - 5\nu_0)(5 - \nu_0)} \right) \qquad [33]$$

where ν_0 is the Poisson ratio of the undamaged grains.

3. Macroscopic modelling of anisotropic damage

3.1. *Representation theorems and expression of the free energy*

Although the choice of the most appropriate tensorial representation for the damage variable is still debated ((Lubarda *et al.*, 1993)), a simple continuous representation of the damage can be adequately obtained by using a second-order symmetric tensor $\underline{\underline{D}}$ as introduced in subsection 2.1.

Since only induced anisotropy is considered, it is assumed that in the undamaged state the material is isotropic elastic linear. As for the isotropic damage modelling, the next step is to choose an expression of the free energy related to the anisotropic elastic damage behavior. A classical result of the representation theorems (invariants theory) is that any scalar valued anisotropic function of vectors and second order tensors, can be expressed as an isotropic function of the original arguments and of the additional tensorial arguments inducing the anisotropy. Thus, the elastic energy W of the damaged body can be represented by an isotropic function of $\underline{\underline{\varepsilon}}$ and the damage tensor $\underline{\underline{D}}$. The most general representation of a polynomial isotropic scalar function of the symmetric tensors $(\underline{\underline{\varepsilon}}, \underline{\underline{D}})$ is a linear combination of the following invariants (see for e.g. (Boehler, 1987)) :

$$\begin{array}{c} tr(\underline{\underline{\varepsilon}}), tr(\underline{\underline{\varepsilon}}^2), tr(\underline{\underline{\varepsilon}}^3), tr(\underline{\underline{D}}), tr(\underline{\underline{D}}^2), tr(\underline{\underline{D}}^3) \\ tr(\underline{\underline{\varepsilon}}.\underline{\underline{D}}), tr(\underline{\underline{\varepsilon}}^2.\underline{\underline{D}}), tr(\underline{\underline{\varepsilon}}.\underline{\underline{D}}^2), tr(\underline{\underline{\varepsilon}}^2.\underline{\underline{D}}^2) \end{array} \qquad [34]$$

Since in the undamaged state ($\underline{\underline{D}} = \underline{\underline{0}}$) or at fixed $\underline{\underline{D}}$ the material is linear elastic, the energy W must be quadratic in $\underline{\underline{\varepsilon}}$. Moreover, if the amplitude of damage is moderate,

2. For the methodology of these methods, the reader can refer to paper (Dormieux *et al.*, 2007), this issue.

it can be assumed that W is affine with $\underline{\underline{D}}$. Thus, the representation of W takes the form :

$$W = \frac{1}{2}[\lambda(tr(\underline{\underline{\varepsilon}}))^2 + 2\mu tr(\underline{\underline{\varepsilon}}^2)] + \quad \frac{1}{2}[\eta tr(\underline{\underline{D}})(tr(\underline{\underline{\varepsilon}}))^2 + \gamma tr(\underline{\underline{D}})tr(\underline{\underline{\varepsilon}}^2)]$$
$$+ \alpha tr(\underline{\underline{\varepsilon}})tr(\underline{\underline{\varepsilon}}.\underline{\underline{D}}) + \chi tr(\underline{\underline{\varepsilon}}^2.\underline{\underline{D}}) \tag{35}$$

α, β, η, and γ are constants which characterize the damage effects on the material behavior. It must be noted that this form of potential has been used by (Murakami *et al.*, 1997) for modelling anisotropic damage of concrete ; similar expression without terms in $tr(\underline{\underline{D}})$ has been also considered by (Dragon *et al.*, 1996).

3.2. *Formulation of the anisotropic damage model*

The macroscopic stress tensor $\underline{\underline{\sigma}}$, obtained by partial derivation of W reads :

$$\underline{\underline{\sigma}} = \frac{\partial W}{\partial \underline{\underline{\varepsilon}}} = \mathbb{D}(\underline{\underline{D}}) : \underline{\underline{\varepsilon}} \tag{36}$$

with

$$\mathbb{D}(\underline{\underline{D}}) = \mathbb{D}_0 + \alpha(\underline{\underline{1}} \otimes \underline{\underline{D}} + \underline{\underline{D}} \otimes \underline{\underline{1}}) + \chi(\underline{\underline{D}}\,\overline{\underline{\otimes}}\,\underline{\underline{1}} + \underline{\underline{1}}\,\overline{\underline{\otimes}}\,\underline{\underline{D}})$$
$$+ \eta\,tr(\underline{\underline{D}})(\underline{\underline{1}} \otimes \underline{\underline{1}}) + \gamma\,tr(\underline{\underline{D}})(\underline{\underline{1}}\,\overline{\underline{\otimes}}\,\underline{\underline{1}}) \tag{37}$$

where $\mathbb{D}_0 = 3k_0\mathbb{J} + 2\mu_0\mathbb{K}$ is still the elastic stiffness of the undamaged material.

The second state law allows to introduce the thermodynamic force associated to the second order tensor $\underline{\underline{D}}$:

$$\underline{\underline{F}}^D = -\frac{\partial W}{\partial \underline{\underline{D}}} \tag{38}$$

The derivation of the damage evolution can be done through the same procedure as for the isotropic damage (subsection 2.3). By analogy with [26] a damage surface, expressed in terms of ^{D} and $\underline{\underline{D}}$, can be choosen in the form :

$$f(\underline{\underline{F}}^D, \underline{\underline{D}}) = \|\underline{\underline{F}}^D\| - (a_0 + a_1 tr(\underline{\underline{D}})). \tag{39}$$

where a_0 and a_1 are material constants : a_0 defines the initial damage threshold while a_1 describe the manner in which the surface evolves with damage.

The evolution of $\underline{\underline{D}}$ is still assumed to follow the normality rule :

$$\dot{\underline{\underline{D}}} = \begin{cases} 0, & \text{if } f < 0 \quad \text{or} \quad f = 0 \text{ and } \dot{f} < 0 \\ \dot{\Lambda}\,\frac{\partial f}{\partial \underline{\underline{F}}^D}, & \text{if } f = 0 \text{ and } \dot{f} = 0 \end{cases} \tag{40}$$

One obtains the damage multiplier Λ (positive scalar) :

$$\dot{\Lambda} = \frac{tr(\underline{\underline{F}}^D \cdot \dot{\underline{\underline{F}}}^D)}{a_1 tr(\underline{\underline{F}}^D)} \qquad [41]$$

4. Isotropic elastic damage behavior with accounts of unilateral effects

In this section, the aim is to extend the isotropic model described in section 2 by incorporating unilateral effects due to microcracks closure. It is well known that the mechanical response of a microcracked medium strongly depends on the opening and closure status of the existing defects in the material. In particular, it is commonly observed that the application of a tensile loading followed by a compression in the same direction leads in the compression regime to a total or partial recovery of the Young modulus. From the modelling point of view, the consideration of the unilateral effects in Continuum Damage Mechanics (CDM) still constitutes a challenging task. The main difficulty lies in the necessity to predict both continuous response of the material and partial or total recovery of the elastic constants during the microcracks closure process. During this microcracks closure, the elastic stiffness must also remain symmetric. A critical review of all these difficulties can be found in (Chaboche, 1992). Several other aspects allowing to point out some spurious energy dissipation are discussed in (Carol *et al.*, 1996). Moreover, (Cormery *et al.*, 2002) discussed for existing models the existence of the thermodynamics potential when unilateral effects are accounted.

We propose here a mathematically rigorous and physically-motivated model based on micromechanical considerations[3]. Again, we assume here that the microcracks are randomly oriented, so that the damage state is isotropic and can be represented by a positive scalar variable ω which may correspond to the microcracks density parameter (see (Budiansky *et al.*, 1976)). In order to account for the unilateral effects, a separated description is proposed when the microcracks are opened or closed by $\mathbb{D}^o(\omega)$ and $\mathbb{D}^{cl}(\omega)$ respectively. The simplest partition of the strain space into two half-spaces is provided by a hyperplan Γ which depends on $\underline{\underline{\varepsilon}}$. It is assumed that the opening and closure state does not depend on the microcracks density parameter but only on the strain state. Therefore, the microcracks are opened if $\Gamma(\underline{\varepsilon}) > 0$ and closed if $\Gamma(\underline{\varepsilon}) \leq 0$. The isotropic stiffness tensor of the damaged material takes then the following form :

$$\mathbb{D}(\omega) = \begin{cases} \mathbb{D}^o(\omega) = 3K^o(\omega)\mathbb{J} + 2\mu^o(\omega)\mathbb{K} & \text{if } \Gamma(\underline{\varepsilon}) > 0 \\ \mathbb{D}^{cl}(\omega) = 3K^{cl}(\omega)\mathbb{J} + 2\mu^{cl}(\omega)\mathbb{K} & \text{if } \Gamma(\underline{\underline{\varepsilon}}) \leq 0 \end{cases} \qquad [42]$$

(K^o, μ^o) and (K^{cl}, μ^{cl}) moduli represent respectively the compressibility and shear moduli for opened and closed microcracks.

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3. As we will see, some micromechanical results can be incorporated in order to enhance the purely macroscopic modelling.

4.1. *Continuity conditions and microcracks closure criterion*

Since the thermodynamic potential has to be continuously differentiable, (K^o, μ^o) and (K^{cl}, μ^{cl}) must fulfill some conditions. Indeed, (Curnier *et al.*, 1995) (see also (Welemane, 2002)) have demonstrated that the elastic energy function W is C^1 continuous if and only if :

$$[\mathbb{D}(\omega)] = \mathbb{D}^o(\omega) - \mathbb{D}^{cl}(\omega) = s(\omega)\frac{\partial \Gamma}{\partial \underline{\underline{\varepsilon}}}(\underline{\underline{\varepsilon}}) \otimes \frac{\partial \Gamma}{\partial \underline{\underline{\varepsilon}}}(\underline{\underline{\varepsilon}}), \forall \underline{\underline{\varepsilon}} | \Gamma(\underline{\underline{\varepsilon}}) = 0 \qquad [43]$$

where s is a continuous scalar valued function depending on ω. We emphasize that $\mathbb{D}^o$ is the stiffness tensor of the damaged materials when cracks are open, while $\mathbb{D}^{cl}$ corresponds to the stiffness when all cracks are closed.

Using [42], it is readily seen that the jump in the stiffness can be expressed as :

$$[\mathbb{D}(\omega)] = 3[K^o(\omega) - K^{cl}(\omega)]\mathbb{J} + 2[\mu^o(\omega) - \mu^{cl}(\omega)]\mathbb{K} \qquad [44]$$

which must be singular and in fact of rank one to satisfy the condition [43] (Curnier *et al.*, 1995). This condition is fulfilled if all determinants of second order obtained from a Voigt representation of $[\mathbb{D}(\omega)]$ are cancelled, that is if :

$$\forall \omega, \mu^o(\omega) = \mu^{cl}(\omega) \qquad [45]$$

The continuity of W is then obtained if the shear moduli does not depend on the microcracks state : $\mu(\omega) = \mu^o(\omega) = \mu^{cl}(\omega)$.

It can be noted that at this step no mathematical conditions are imposed on the compressibility moduli.

4.2. *Moduli recovery conditions*

Inspired by some micromechanical considerations on the progressive closure effects of microcracks, we assume as well that the compressibility modulus corresponding to all closed microcracks takes the initial value K_0 of the sound material. Therefore, the modulus restitution condition reads :

$$K^{cl}(\omega) = K_0 \qquad [46]$$

and we will denote $K^o(\omega) = K(\omega)$.

At this level of development, it remains to precise the expression of the function Γ which defines the microcracks closure criterion. In view of [44] and [45], the tensor $[\mathbb{D}(\omega)]$ is written in the form :

$$[\mathbb{D}(\omega)] = 3[K^o(\omega) - K^{cl}(\omega)]\mathbb{J} = [K^o(\omega) - K^{cl}(\omega)]\underline{\underline{1}} \otimes \underline{\underline{1}} \qquad [47]$$

Comparing [43] and [47], it follows that the closure/opening criterion reads :

$$\Gamma(\underline{\underline{\varepsilon}}) = tr(\underline{\underline{\varepsilon}}) = \underline{0} \qquad [48]$$

It must be emphasized that the simplicity of the microcrack opening-closure transition criterion is due to the isotropy of the damage. In the case of a anisotropic damage, the methodology followed in this work would lead to a more complex criterion which depends in particular on the microcracks orientation.

4.3. *Final expression of the thermodynamic potential*

Finally, the thermodynamic potential takes the following form :

$$W(\underline{\underline{\varepsilon}}, \omega) = \frac{1}{2}\underline{\underline{\varepsilon}} : \mathbb{D}(\omega) : \underline{\underline{\varepsilon}}, \; \mathbb{D}(\omega) = \left\{ \begin{array}{ll} 3K(\omega)\mathbb{J} + 2\mu(\omega)\mathbb{K} & \text{if } tr(\underline{\underline{\varepsilon}}) > 0 \\ 3K_0\mathbb{J} + 2\mu(\omega)\mathbb{K} & \text{if } tr(\underline{\underline{\varepsilon}}) \leq 0 \end{array} \right. \quad [49]$$

In order to provide a physical basis to the damage model, we consider as before for $K(\omega)$ and $\mu(\omega)$ micromechanical results established by (Ponte-Castaneda *et al.*, 1995) for microcracked media [32].

For the derivation of the incremental formulation we adopt the same loading function and follow the same procedure as in section 2. The main difference is that we have here two different expression of the thermodynamics potential according to the fact that damage is activated or not. It follows the rate form of the stress tensor for the elastic unilateral damage law :

$$\dot{\underline{\underline{\sigma}}} = \mathbb{L} : \dot{\underline{\underline{\varepsilon}}} \qquad [50]$$

with, $\underline{\underline{e}}$ being the deviatoric part of $\underline{\underline{\varepsilon}}$, the tangent operator $\mathbb{L}$ given by :

$$\mathbb{L} = 3k_1\mathbb{J} + 2k_2\mathbb{K} - 2k_3(\underline{\underline{1}} \otimes \underline{\underline{e}} + \underline{\underline{e}} \otimes \underline{\underline{1}}) - 2k_4\underline{\underline{e}} \otimes \underline{\underline{e}} \qquad [51]$$

$$k_2 = \mu(\omega), \, k_4 = \frac{2(\mu'(\omega))^2}{h},$$

$$\left\{ \begin{array}{llll} k_1 = K(\omega) - H_1 & \text{and} & k_3 = K(\omega) - H_3 & \text{if } tr(\underline{\underline{\varepsilon}}) > 0 \\ k_1 = K_0 & \text{and} & k_3 = 0 & \text{if } tr(\underline{\underline{\varepsilon}}) \leq 0 \end{array} \right. \qquad [52]$$

with $H_1 = \dfrac{(K'(\omega)tr\underline{\underline{\varepsilon}})^2}{h}$ and $H_3 = \dfrac{\mu'(\omega)K'(\omega)}{h}$. The quantity h is expressed as : $h = H_0\eta + K''(\omega)tr(\underline{\underline{\varepsilon}})\underline{\underline{1}} + \mu''(\omega)\underline{\underline{e}}$ where $K''(\omega)$ and $\mu''(\omega)$ are the second derivative of $K(\omega)$ and $\mu(\omega)$ respectively whereas $K'(\omega)$ and $\mu'(\omega)$ are the first derivative of $K(\omega)$ and $\mu(\omega)$ respectively.

Different versions of the isotropic damage models can be obtained by considering, as previously, expressions of $K(\omega)$ and $\mu(\omega)$ provided by micromechanics (see again (Dormieux *et al.*, 2007)). Obviously, recalling that we refer here to a purely macroscopic approach, any suitable function can be considered.

5. Conclusions

This chapter is devoted to a general presentation of the basic principles of Continuum Damage Mechanics and the associated thermodynamics framework. It may also provide a general view of the classical procedure used for isotropic damage. In order to take into account the damage-induced anisotropy, representation theorems appear as a useful tool ; however, it must be pointed out that the approach presented here is limited to moderate damage and there is need to go beyond this hypothesis. A suitable modelling of the *unilateral* isotropic damage is also proposed by imposing some continuity conditions as well as the condition of elastic moduli recovery at damage deactivation. This approach for the unilateral damage modelling has been succesfully developed by (Welemane, 2002) (see also (Welemane *et al.*, 2003)) in the context of anisotropic damage. Obviously, there other important topics which are not adressed in this introductory paper : Poroelastic damage (Dormieux *et al.*, 2006), coupling between plasticity and damage, effect of initial anisotropy, friction on closed cracks faces, etc.).

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