Module: 7 Torsion of Prismatic Bars

7.2.1 TORSION OF ELLIPTICAL CROSS-SECTION

Let the warping function is given by

$$\psi = Axy \tag{7.15}$$

where A is a constant. This also satisfies the Laplace equation. The boundary condition gives

$$(Ay - y)\frac{dy}{dS} - (Ax + x)\frac{dx}{dS} = 0$$

or $\quad y(A-1)\dfrac{dy}{dS} - x(A+1)\dfrac{dx}{dS} = 0$

i.e., $(A+1)2x\dfrac{dx}{dS} - (A-1)2y\dfrac{dy}{dS} = 0$

or $\quad \dfrac{d}{dS}[(A+1)x^2 - (A-1)y^2] = 0$

Integrating, we get

$(1+A)x^2+(1-A)y^2$ = constant.

This is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These two are identical if

$$\frac{a^2}{b^2} = \frac{1-A}{1+A}$$

or $A = \dfrac{b^2 - a^2}{b^2 + a^2}$

Therefore, the function given by

$$\psi = \frac{b^2 - a^2}{b^2 + a^2}xy \tag{7.16}$$

represents the warping function for an elliptic cylinder with semi-axes a and b under torsion. The value of polar moment of inertia J is

$$J = \iint (x^2 + y^2 + Ax^2 - Ay^2)dxdy \tag{7.17}$$

$$= (A+1)\int\int x^2 dxdy + (1-A)\int\int y^2 dxdy$$

$$J = (A+1)I_y + (1-A)I_x \qquad (7.18)$$

where $I_x = \dfrac{\pi ab^3}{4}$ and $I_y = \dfrac{\pi a^3 b}{4}$

Substituting the above values in (7.18), we obtain

$$J = \frac{\pi a^3 b^3}{a^2 + b^2}$$

But $\theta = \dfrac{M_t}{GI_P} = \dfrac{M_t}{GJ}$

Therefore, $M_t = GJ\theta$

$$= G\theta \frac{\pi a^3 b^3}{a^2 + b^2}$$

or $$\theta = \frac{M_t}{G}\frac{a^2 + b^2}{\pi a^3 b^3}$$

The shearing stresses are given by

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

$$= M_t \frac{a^2 + b^2}{\pi a^3 b^3}\left(\frac{b^2 - a^2}{b^2 + a^2} + 1\right)x$$

or $\tau_{yz} = \dfrac{2M_t x}{\pi a^3 b}$

Similarly, $\tau_{xz} = \dfrac{2M_t y}{\pi ab^3}$

Therefore, the resultant shearing stress at any point (x, y) is

$$\tau = \sqrt{\tau_{yz}^2 + \tau_{xz}^2} = \frac{2M_t}{\pi a^3 b^3}\left[b^4 x^2 + a^4 y^2\right]^{\frac{1}{2}} \qquad (7.19)$$

Determination of Maximum Shear Stress

To determine where the maximum shear stress occurs, substitute for x^2 from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or $x^2 = a^2(1 - y^2/b^2)$

and $\tau = \frac{2M_t}{\pi a^3 b^3}\left[a^2 b^4 + a^2(a^2 - b^2)y^2\right]^{\frac{1}{2}}$

Since all terms under the radical (power 1/2) are positive, the maximum shear stress occurs when y is maximum, i.e., when $y = b$. Thus, maximum shear stress τ_{max} occurs at the ends of the minor axis and its value is

$$\tau_{max} = \frac{2M_t}{\pi a^3 b^3}(a^4 b^2)^{1/2}$$

Therefore, $\tau_{max} = \frac{2M_t}{\pi a b^2}$ (7.20)

For $a = b$, this formula coincides with the well-known formula for circular cross-section. Knowing the warping function, the displacement w can be easily determined.

Therefore, $w = \theta\psi = \frac{M_t(b^2 - a^2)}{\pi a^3 b^3 G}xy$ (7.21)

The contour lines giving w = constant are the hyperbolas shown in the Figure 7.4 having the principal axes of the ellipse as asymptotes.

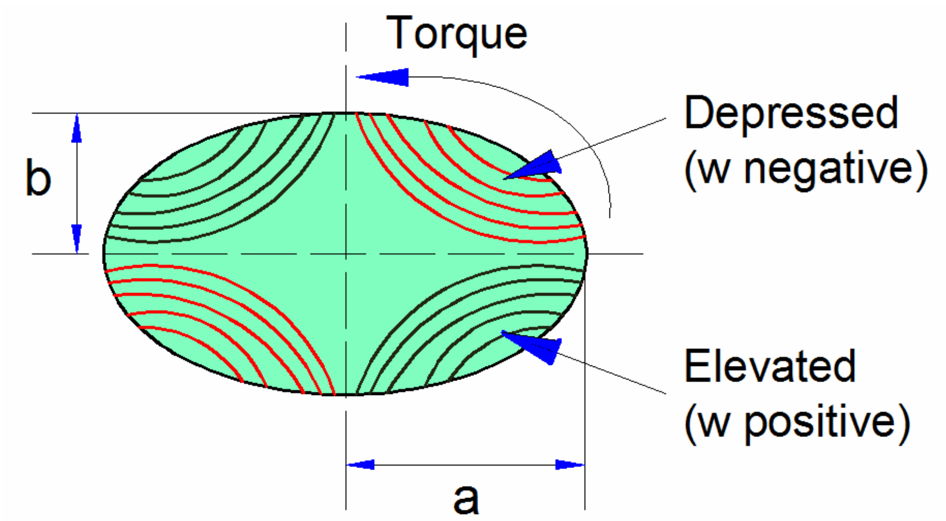


Figure 7.4 Cross-section of elliptic bar and contour lines of *w*

7.2.2 PRANDTL'S MEMBRANE ANALOGY

It becomes evident that for bars with more complicated cross-sectional shapes, more analytical solutions are involved and hence become difficult. In such situations, it is desirable to use other techniques – experimental or otherwise. The membrane analogy introduced by Prandtl has proved very valuable in this regard.

Let a thin homogeneous membrane, like a thin rubber sheet be stretched with uniform tension fixed at it's edge which is a given curve (the cross-section of the shaft) in the xy-plane as shown in the figure 7.5.

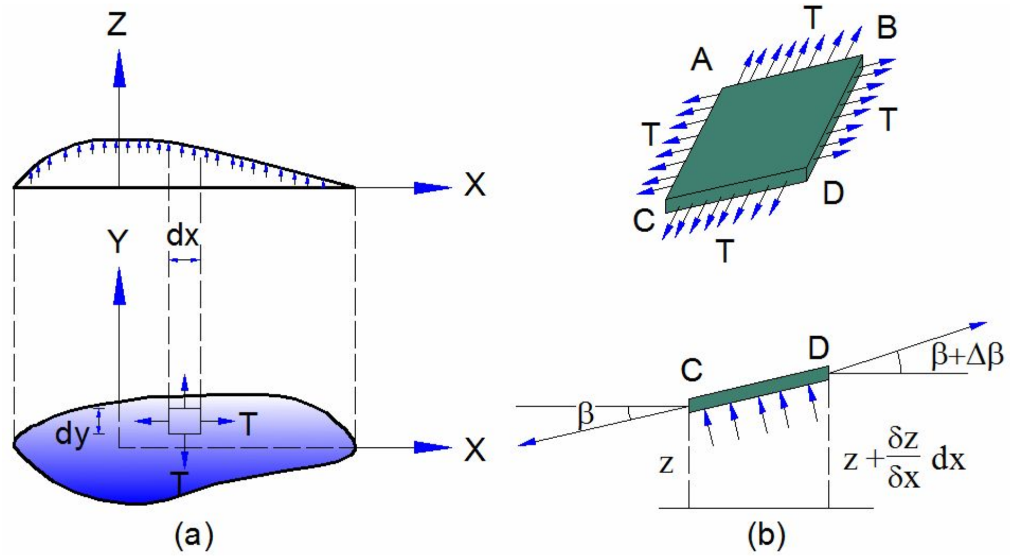


Figure 7.5 Stretching of a membrane

When the membrane is subjected to a uniform lateral pressure p, it undergoes a small displacement z where z is a function of x and y.

Consider the equilibrium of an infinitesimal element ABCD of the membrane after deformation. Let F be the uniform tension per unit length of the membrane. The value of the initial tension F is large enough to ignore its change when the membrane is blown up by the small pressure p. On the face AD, the force acting is $F.dy$. This is inclined at an angle β to the x-axis. Also, tan β is the slope of the face AB and is equal to $\frac{\partial z}{\partial x}$. Hence the component of Fdy in z-direction is $\left(-Fdy\frac{\partial z}{\partial x}\right)$. The force on face BC is also Fdy but is inclined at an angle $(\beta + \Delta\beta)$ to the x-axis. Its slope is, therefore,

$$\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx$$

and the component of the force in the z-direction is

$$Fdy\left[\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx\right]$$

Similarly, the components of the forces Fdx acting on face AB and CD are

$-Fdx\frac{\partial z}{\partial y}$ and $Fdx\left[\frac{\partial z}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)dy\right]$

Therefore, the resultant force in z-direction due to tension F

$$= -Fdy\frac{\partial z}{\partial x}+Fdy\left[\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}dx\right]-Fdx\frac{\partial z}{\partial y}+Fdx\left[\frac{\partial z}{\partial y}+\frac{\partial^2 z}{\partial y^2}dy\right]$$

$$= F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)dxdy$$

But the force p acting upward on the membrane element ABCD is $p\ dxdy$, assuming that the membrane deflection is small.

Hence, for equilibrium,

$$F\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right) = -p$$

or $$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = -p/F \qquad (7.22)$$

Now, if the membrane tension F or the air pressure p is adjusted in such a way that p/F becomes numerically equal to $2G\theta$, then Equation (7.22) of the membrane becomes identical to Equation (7.8) of the torsion stress function ϕ. Further if the membrane height z remains zero at the boundary contour of the section, then the height z of the membrane becomes numerically equal to the torsion stress function $\phi = 0$. The slopes of the membrane are then equal to the shear stresses and these are in a direction perpendicular to that of the slope.

Further, the twisting moment is numerically equivalent to twice the volume under the membrane [Equation (7.14)].

Table 7.1 Analogy between Torsion and Membrane Problems

Membrane problem	Torsion Problem
Z	ϕ
$\frac{1}{S}$	G
P	2θ
$-\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$	τ_{zy},τ_{zx}
2 (volume beneath membrane)	M_t

The membrane analogy provides a useful experimental technique. It also serves as the basis for obtaining approximate analytical solutions for bars of narrow cross-section as well as for member of open thin walled section.

7.2.3 TORSION OF THIN-WALLED SECTIONS

Consider a thin-walled tube subjected to torsion. The thickness of the tube may not be uniform as shown in the Figure 7.6.

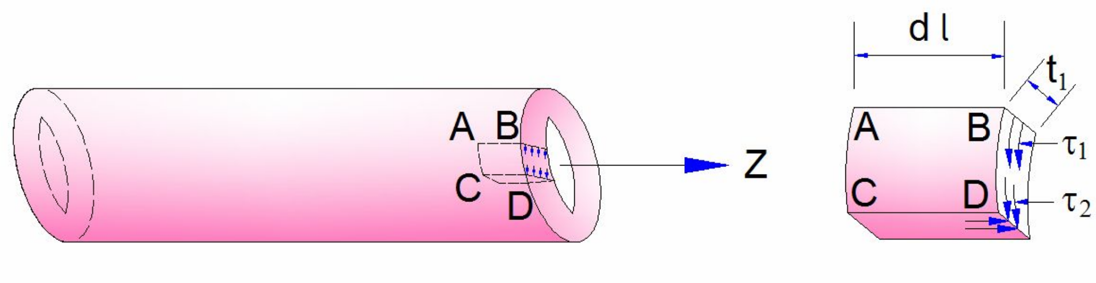


Figure 7.6 Torsion of thin walled sections

Since the thickness is small and the boundaries are free, the shear stresses will be essentially parallel to the boundary. Let τ be the magnitude of shear stress and t is the thickness.

Now, consider the equilibrium of an element of length Δl as shown in Figure 7.6. The areas of cut faces *AB* and *CD* are $t_1\,\Delta l$ and $t_2\,\Delta l$ respectively. The shear stresses (complementary shears) are τ_1 and τ_2.

For equilibrium in *z*-direction, we have

$$-\tau_1 t_1 \Delta l + \tau_2 t_2 \Delta l = 0$$

Therefore, $\tau_1 t_1 = \tau_2 t_2 = q$ = constant

Hence the quantity $\tau\, t$ is constant. This is called the shear flow q, since the equation is similar to the flow of an incompressible liquid in a tube of varying area.

Determination of Torque Due to Shear and Rotation

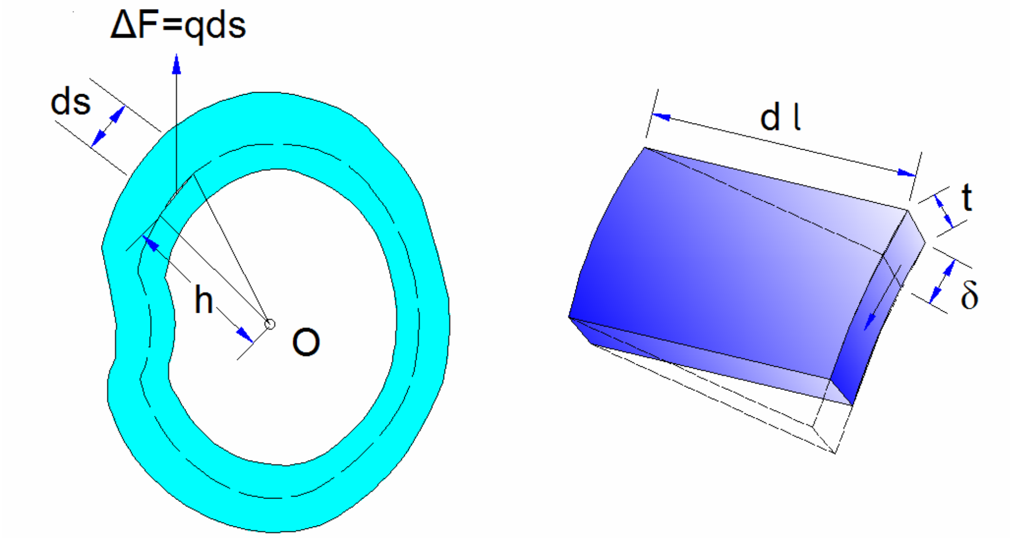


Figure 7.7 Cross section of a thin-walled tube and torque due to shear

Consider the torque of the shear about point O (Figure 7.7).
The force acting on the elementary length dS of the tube = $\Delta F = \tau t\, dS = q\, dS$
The moment arm about O is h and hence the torque = $\Delta M_t = (qdS)\, h$
Therefore, $\Delta M_t = 2qdA$

where dA is the area of the triangle enclosed at O by the base dS.

Hence the total torque is

$M_t = \Sigma\, 2qdA +$

Therefore, $M_t = 2qA$ (7.23)

where A is the area enclosed by the centre line of the tube. Equation (7.23) is generally known as the "Bredt-Batho" formula.

To Determine the Twist of the Tube

In order to determine the twist of the tube, Castigliano's theorem is used. Referring to Figure 7.7(b), the shear force on the element is $\tau\, t\, dS = qdS$. Due to shear strain γ, the force does work equal to ΔU

i.e., $\Delta U = \frac{1}{2}(\tau\, tdS)\delta$

$$= \frac{1}{2}(\tau\, tdS)\gamma . \Delta l$$

$$= \frac{1}{2}(\tau\, t dS).\Delta l.\frac{\tau}{G} \text{ (since } \tau = G\gamma \text{)}$$

$$= \frac{\tau^2 t^2 dS \Delta l}{2Gt}$$

$$= \frac{q^2 dS \Delta l}{2Gt}$$

$$= \frac{q^2 \Delta l}{2G}.\frac{dS}{t}$$

$$\Delta U = \frac{M_t^2 \Delta l}{8A^2 G}.\frac{dS}{t}$$

Therefore, the total elastic strain energy is

$$U = \frac{M_t^2 \Delta l}{8A^2 G}\oint \frac{dS}{t}$$

Hence, the twist or the rotation per unit length (Δl = 1) is

$$\theta = \frac{\partial U}{\partial M_t} = \frac{M_t}{4A^2 G}\oint \frac{dS}{t}$$

or $$\theta = \frac{2qA}{4A^2 G}\oint \frac{dS}{t}$$

or $$\theta = \frac{q}{2AG}\oint \frac{dS}{t} \qquad (7.24)$$

7.2.4 TORSION OF THIN-WALLED MULTIPLE-CELL CLOSED SECTIONS

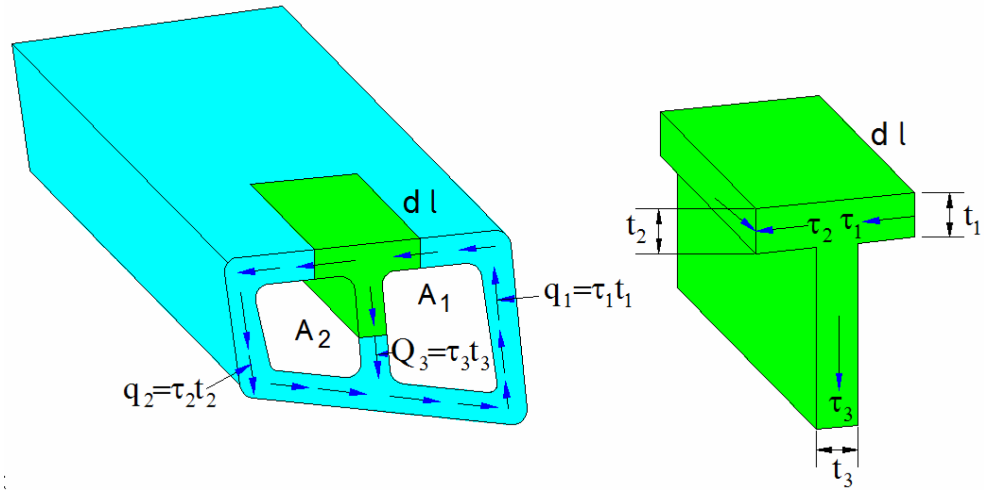


Figure 7.8 Torsion of thin-walled multiple cell closed section

Consider the two-cell section shown in the Figure 7.8. Let A_1 and A_2 be the areas of the cells 1 and 2 respectively. Consider the equilibrium of an element at the junction as shown in the Figure 7.8(b). In the direction of the axis of the tube, we can write

$$-\tau_1 t_1 \Delta l + \tau_2 t_2 \Delta l + \tau_3 t_3 \Delta l = 0$$

or $\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3$

i.e., $q_1 = q_2 + q_3$

This is again equivalent to a fluid flow dividing itself into two streams. Now, choose moment axis, such as point O as shown in the Figure 7.9.

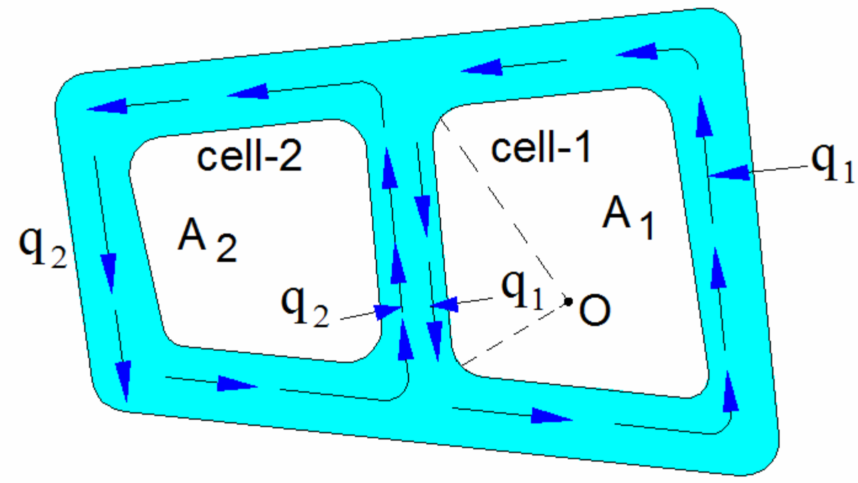


Figure. 7.9 Section of a thin walled multiple cell beam and moment axis

The shear flow in the web is considered to be made of q_1 and $-q_2$, since $q_3 = q_1 - q_2$.
Moment about O due to q_1 flowing in cell 1 (including web) is

$$M_{t_1} = 2q_1A_1$$

Similarly, the moment about O due to q_2 flowing in cell 2 (including web) is

$$M_{t_2} = 2q_2(A_2+A_1) - 2q_2A_1$$

The second term with the negative sign on the right hand side is the moment due to shear flow q_2 in the middle web.

Therefore, The total torque is

$$M_t = M_{t_1} + M_{t_2}$$

$$M_t = 2q_1A_1 + 2q_2A_2 \qquad \text{(a)}$$

To Find the Twist (θ)

For continuity, the twist of each cell should be the same.

We have

$$\theta = \frac{q}{2AG}\oint \frac{dS}{t}$$

or $$2G\theta = \frac{1}{A}\int \frac{qdS}{t}$$

Let $a_1 = \oint \frac{dS}{t}$ for Cell 1 including the web

$a_2 = \oint \frac{dS}{t}$ for Cell 2 including the web

$a_{12} = \oint \frac{dS}{t}$ for the web only

Then for Cell 1

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2) \qquad \text{(b)}$$

For Cell 2

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1) \qquad \text{(c)}$$

Equations (a), (b) and (c) are sufficient to solve for q_1, q_2 and θ.

7.2.5 NUMERICAL EXAMPLES

Example 7.1

A hollow aluminum tube of rectangular cross-section shown in Figure below, is subjected to a torque of 56,500 *m-N* along its longitudinal axis. Determine the shearing stresses and the angle of twist. Assume *G* = 27.6x10^9 N/m^2.

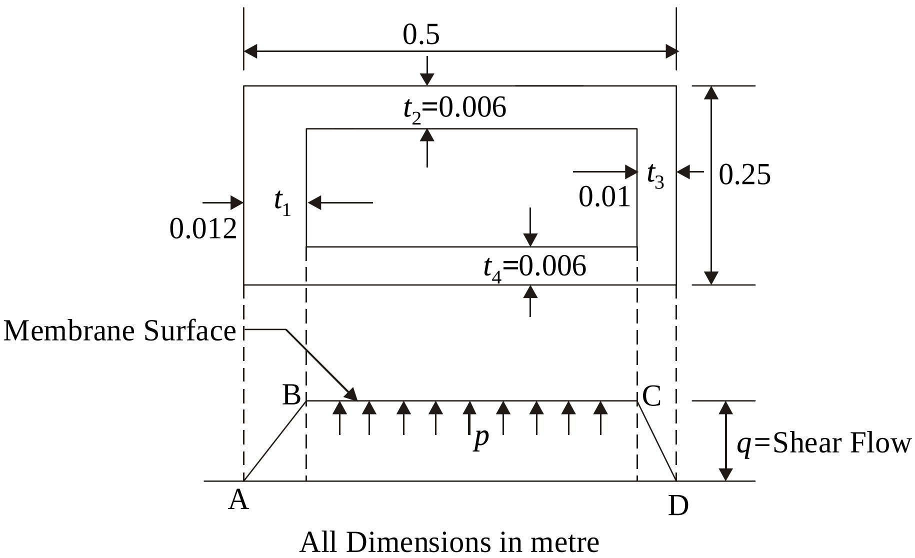


Figure 7.10

Solution: The above figure shows the membrane surface *ABCD*

Now, the Applied torque $= M_t = 2qA$

$$56{,}500 = 2q(0.5\text{x}0.25)$$

$$56{,}500 = 0.25q$$

hence, q = 226000 *N/m*.

Now, the shearing stresses are

$$\tau_1 = \frac{q}{t_1} = \frac{226000}{0.012} = 18.833 \times 10^6 N/m^2$$

$$\tau_2 = \frac{q}{t_2} = \frac{226000}{0.006} = 37.667 \times 10^6 N/m^2$$

$$\tau_3 = \frac{226000}{0.01} = 22.6 \times 10^6 N/m^2$$

Now, the angle of twist per unit length is

$$\theta = \frac{q}{2GA}\oint\frac{ds}{t}$$

Therefore,

$$\theta = \frac{226000}{2\text{x}27.6\text{x}10^9\text{x}0.125}\left[\frac{0.25}{0.012} + \frac{0.5}{0.006}(2) + \frac{0.25}{0.01}\right]$$

or θ = 0.00696014 *rad/m*

Example 7.2

The figure below shows a two-cell tubular section as formed by a conventional airfoil shape, and having one interior web. An external torque of 10,000 *Nm* is acting in a clockwise direction. Determine the internal shear flow distribution. The cell areas are as follows:

$\boldsymbol{A_1}$ **= 680 *cm*^{2}** $\boldsymbol{A_2}$ **= 2000 *cm*^{2}**

The peripheral lengths are indicated in Figure

***Solution*:**

For Cell 1, $a_1 = \oint\frac{dS}{t}$ (including the web)

$$= \frac{67}{0.06} + \frac{33}{0.09}$$

therefore, a_1 = 148.3

For Cell 2,

$$a_2 = \frac{33}{0.09} + \frac{63}{0.09} + \frac{48}{0.09} + \frac{67}{0.08}$$

Therefore, $a_2 = 2409$

For web,

$$a_{12} = \frac{33}{0.09} = 366$$

Now, for Cell 1,

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$$

$$= \frac{1}{680}(1483 q_1 - 366 q_2)$$

Therefore, $2G\theta = 2.189 q_1 - 0.54 q_2$ (i)

For Cell 2,

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$$

$$= \frac{1}{2000}(2409 q_2 - 366 q_1)$$

Therefore, $2G\theta = 1.20 q_2 - 0.18 q_1$ (ii)

Equating (i) and (ii), we get

$2.18\, q_1 - 0.54 q_2 = 1.20 q_2 - 0.18 q_1$

or $2.36 q_1 - 1.74 q_2 = 0$

or $q_2 = 1.36 q_1$

The torque due to shear flows should be equal to the applied torque

Hence, from Equation (a),

$M_t = 2q_1 A_1 + 2q_2 A_2$

$10{,}000 \times 100 = 2q_1 \times 680 + 2q_2 \times 2000$

$= 1360 q_1 + 4000 q_2$

Substituting for q_2, we get

$10000 \times 100 = 1360 q_1 + 4000 \times 1.36 q_1$

Therefore,
q_1 = 147 N and q_2 = 200 N

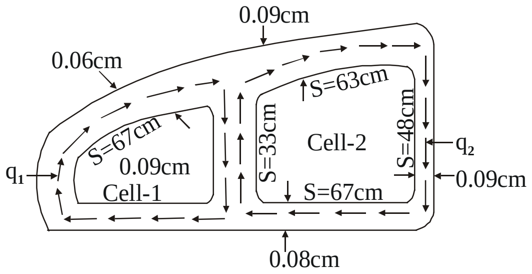


Figure 7.11

Example 7.3
A thin walled steel section shown in figure is subjected to a twisting moment *T*. Calculate the shear stresses in the walls and the angle of twist per unit length of the box.

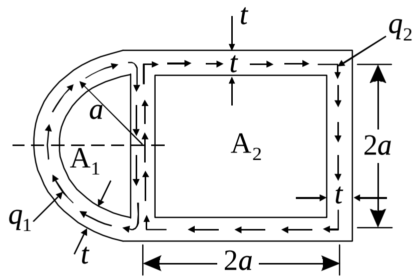


Figure 7.12

Solution: Let A_1 and A_2 be the areas of the cells (1) and (2) respectively.

$$\therefore A_1 = \frac{\pi a^2}{2}$$

$$A_2 = (2a \times 2a) = 4a^2$$

For Cell (1),

$$a_1 = \oint \frac{ds}{t} \text{ (Including the web)}$$

$$a_1 = \left(\frac{\pi a + 2a}{t}\right)$$

For Cell (2),

$$a_2 = \oint \frac{ds}{t}$$

$$= \frac{2a}{t} + \frac{2a}{t} + \frac{2a}{t} + \frac{2a}{t}$$

$$\therefore a_2 = \left(\frac{8a}{t}\right)$$

For web,

$$a_{12} = \left(\frac{2a}{t}\right)$$

Now,
For Cell (1),

$$2G\theta = \frac{1}{A_1}\left(a_1 q_1 - a_{12} q_2\right)$$

$$= \frac{2}{\pi a^2}\left[\frac{(\pi a + 2a)}{t} q_1 - \left(\frac{2a}{t}\right) q_2\right]$$

$$= \frac{2a}{\pi t a^2}\left[(2 + \pi) q_1 - 2 q_2\right]$$

$$\therefore 2G\theta = \frac{2}{\pi a t}\left[(\pi + 2) q_1 - 2 q_2\right] \qquad (1)$$

For Cell (2),

$$2G\theta = \frac{1}{A_2}\left(a_2 q_2 - a_{12} q_1\right)$$

$$= \frac{1}{4a^2}\left[\frac{8a}{t} q_2 - \frac{2a}{t} q_1\right]$$

$$= \frac{2a}{4a^2 t}\left[4 q_2 - q_1\right]$$

$$\therefore 2G\theta = \frac{1}{2at}\left[4 q_2 - q_1\right] \qquad (2)$$

Equating (1) and (2), we get,

$$\frac{2}{\pi a t}\left[(\pi + 2) q_1 - 2 q_2\right] = \frac{1}{2at}\left[4 q_2 - q_1\right]$$

or $\dfrac{2}{\pi}\left[(\pi + 2) q_1 - 2 q_2\right] = \dfrac{1}{2}\left[4 q_2 - q_1\right]$

$$\frac{4}{\pi}\left[(\pi+2)q_1 - 2q_2\right] = \left[4q_2 - q_1\right]$$

$$\therefore \frac{4(\pi+2)}{\pi}q_1 - \frac{8}{\pi}q_2 - 4q_2 + q_1 = 0$$

$$\left[\frac{4(\pi+2)}{\pi} + 1\right]q_1 - \left[\frac{8}{\pi} + 4\right]q_2 = 0$$

$$\left[\frac{4(\pi+2)+\pi}{\pi}\right]q_1 - \left[\frac{8+4\pi}{\pi}\right]q_2 = 0$$

or $(4\pi + 8 + \pi)q_1 = (8 + 4\pi)q_2$

$$\therefore q_2 = \left(\frac{5\pi+8}{4\pi+8}\right)q_1$$

But the torque due to shear flows should be equal to the applied torque.

i.e., $T = 2q_1A_1 + 2q_2A_2$ (3)

Substituting the values of q_2, A_1 and A_2 in (3), we get,

$$T = 2q_1\left(\frac{\pi a^2}{2}\right) + 2\left(\frac{5\pi+8}{4\pi+8}\right)q_1.4a^2$$

$$= \pi a^2 q_1 + 8a^2\left(\frac{5\pi+8}{4\pi+8}\right)q_1$$

$$\therefore T = \left[\frac{a^2(\pi^2 + 12\pi + 16)}{(\pi+2)}\right]q_1$$

$$\therefore q_1 = \frac{(\pi+2)T}{a^2(\pi^2 + 12\pi + 16)}$$

Now, from equation (1), we have,

$$2G\theta = \frac{2}{\pi a t}\left[(\pi+2)\frac{(\pi+2)T}{a^2(\pi^2+12\pi+16)} - 2\left(\frac{5\pi+8}{4\pi+8}\right)\frac{(\pi+2)T}{a^2(\pi^2+12\pi+16)}\right]$$

Simplifying, we get the twist as $\theta = \left[\frac{(2\pi+3)T}{2Ga^3t(\pi^2+12\pi+16)}\right]$

Example 7.4

A thin walled box section having dimensions $2a \times a \times t$ is to be compared with a solid circular section of diameter as shown in the figure. Determine the thickness *t* so that the two sections have:

(a) Same maximum shear stress for the same torque.
(b) The same stiffness.

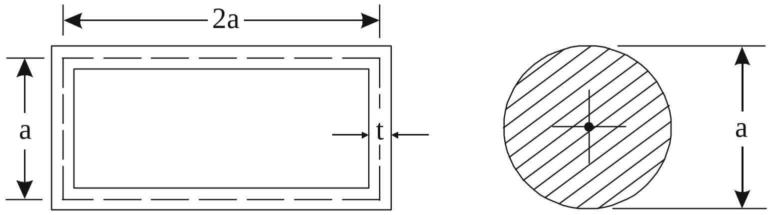


Figure 7.13

Solution: (a) For the box section, we have

$$T = 2qA$$

$$= 2.\tau.t.A$$

$$T = 2.\tau.t.2a \times a$$

$$\therefore \tau = \frac{T}{4a^2 t} \qquad (a)$$

Now, For solid circular section, we have

$$\frac{T}{I_p} = \frac{\tau}{r}$$

Where I_p = Polar moment of inertia

$$\therefore \frac{T}{\left(\frac{\pi a^4}{32}\right)} = \frac{\tau}{\left(\frac{a}{2}\right)}$$

$$or \frac{32T}{\pi a^4} = \frac{2\tau}{a}$$

$$\therefore \tau = \left(\frac{16T}{\pi a^3}\right) \qquad (b)$$

Equating (a) and (b), we get

$$\frac{T}{4a^2 t} = \frac{16T}{\pi a^3} \qquad \therefore 64a^2 tT = \pi a^3 T$$

$$\therefore t = \frac{\pi a}{64}$$

(b) The stiffness of the box section is given by

$$\theta = \frac{q}{2GA}\oint \frac{ds}{t}$$

Here $T = 2qA$ $\quad\therefore q = \frac{T}{2A}$

$$\therefore \theta = \frac{T}{4GA^2}\left[\frac{a}{t}+\frac{2a}{t}+\frac{a}{t}+\frac{2a}{t}\right]$$

$$= \frac{6aT}{4GA^2 t}$$

$$= \frac{6aT}{4G\left(2a^2\right)^2 t}$$

$$\therefore \theta = \frac{6aT}{16a^4 Gt} \qquad (c)$$

The stiffness of the Solid Circular Section is

$$\theta = \frac{T}{GI_p} = \frac{T}{G\left(\frac{\pi a^4}{32}\right)} = \frac{32T}{G\pi a^4} \qquad (d)$$

Equating (c) and (d), we get

$$\frac{6aT}{16a^4 Gt} = \frac{32T}{G\pi a^4}$$

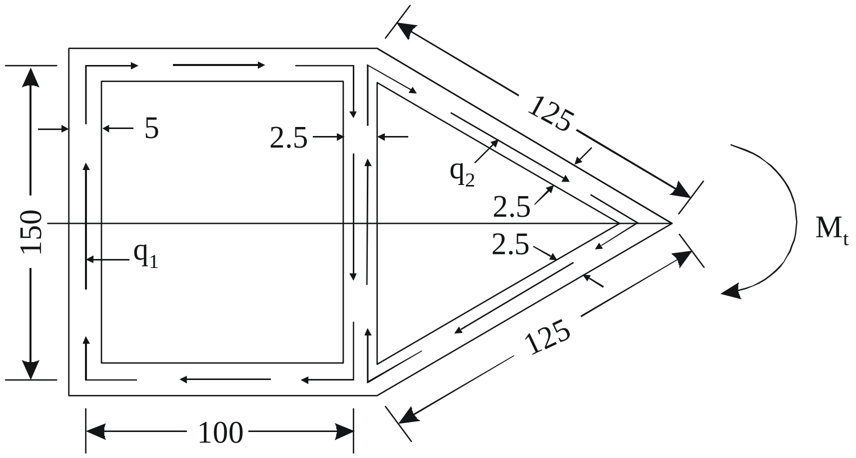
$$\frac{6a}{16t} = \frac{32}{\pi}$$

$$\therefore t = \frac{6\pi a}{16\times 32}$$

$$\therefore t = \frac{3}{4}\left(\frac{\pi a}{64}\right)$$

Example 7.5

A two-cell tube as shown in the figure is subjected to a torque of 10*kN-m*. Determine the Shear Stress in each part and angle of twist per metre length. Take modulus of rigidity of the material as 83 *kN/mm²*.



All dimensions in mm

Figure 7.14

***Solution*:** For Cell 1

Area of the Cell = $A_1 = 150 \times 100 = 15000 mm^2$

$$a_1 = \oint \frac{ds}{t} \text{ (including web)}$$

$$= \frac{150}{5} + \frac{100}{5} + \frac{150}{2.5} + \frac{100}{5}$$

$$= 130$$

For Cell 2

Area of the cell = $A_2 = \frac{1}{2} \times 150 \times \sqrt{(125)^2 - (75)^2}$

$= 7500 mm^2$

$$\therefore a_2 = \oint \frac{ds}{t} \text{ (including web)}$$

$$= \frac{150}{2.5} + \frac{125}{2.5} + \frac{125}{2.5}$$

$$\therefore a_2 = 160$$

For the web,

$$a_{12} = \frac{150}{2.5} = 60$$

For Cell (1)

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$$

$$\therefore 2G\theta = \frac{1}{15000}(130q_1 - 60q_2) \qquad (a)$$

For Cell (2)

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$$

$$= \frac{1}{7500}(160q_2 - 60q_1) \qquad (b)$$

Equating (a) and (b), we get

$$\frac{1}{15000}(130q_1 - 60q_2) = \frac{1}{7500}(160q_2 - 60q_1)$$

Solving, $q_1 = 1.52q_2$ $\qquad (c)$

Now, the torque due to shear flows should be equal to the applied torque.

i.e., $M_t = 2q_1 A_1 + 2q_2 A_2$

$$10 \times 10^6 = 2q_1(15000) + 2q_2(7500) \qquad (d)$$

Substituting (c) in (d), we get

$$10 \times 10^6 = 2 \times 15000(1.52q_2) + 2q_2(7500)$$

$$\therefore q_2 = 165.02N$$

$$\therefore q_1 = 1.52 \times 165.02 = 250.83N$$

Shear flow in the web = $q_3 = (q_1 - q_2) = (250.83 - 165.02)$

$$\therefore q_3 = 85.81N$$

$$\therefore \tau_1 = \frac{q_1}{t_1} = \frac{250.83}{5} = 50.17N/mm^2$$

$$\tau_2 = \frac{q_2}{t_2} = \frac{165.02}{2.5} = 66.01N/mm^2$$

$$\tau_3 = \frac{q_3}{t_3} = \frac{85.81}{2.5} = 34.32N/mm^2$$

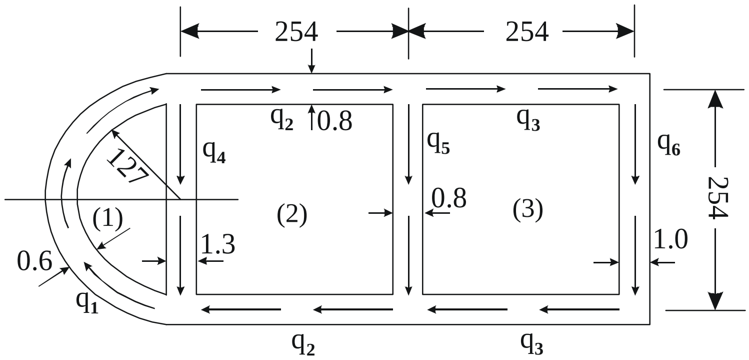
Now, the twist θ is computed by substituting the values of q_1 and q_2 in equation (a) i.e., $2G\theta = \frac{1}{15000}\left[130 \times 250.83 \times 60 \times 165.02\right]$

$\therefore \theta = \frac{1}{15000} \times \frac{22706.7}{83 \times 1000} = 1.824 \times 10^{-5}\ radians/mm\ length$

or $\theta = 1.04$ *degrees/m length*

Example 7.6

A tubular section having three cells as shown in the figure is subjected to a torque of 113 *kN-m*. Determine the shear stresses developed in the walls of the section.



All dimensions in mm

Figure 7.15

Solution: Let $q_1, q_2, q_3, q_4, q_5, q_6$ be the shear flows in the various walls of the tube as shown in the figure. $A_1, A_2, and\ A_3$ be the areas of the three cells.

$\therefore A_1 = \frac{\pi}{2}(127)^2 = 25322 mm^2$

$A_2 = 254 \times 254 = 64516 mm^2$

$A_3 = 64516 mm^2$

Now, From the figure,

$q_1 = q_2 + q_4$

$q_2 = q_3 + q_5$

$q_3 = q_6$

or $q_1 = \tau_1 t_1 = \tau_2 t_2 + \tau_4 t_4$

$q_2 = \tau_2 t_2 = \tau_3 t_3 + \tau_5 t_5$ (1)

$q_3 = \tau_3 t_3 = \tau_6 t_6$

Where $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\ and\ \tau_6$ are the Shear Stresses in the various walls of the tube.

Now, The applied torque is

$$M_t = 2A_1q_1 + 2A_2q_2 + 2A_3q_3$$
$$= 2(A_1\tau_1t_1 + A_2\tau_2t_2 + A_3\tau_3t_3)$$

i.e., $113\times10^6 = 2[(25322\tau_1\times0.8)+(64516\tau_2\times0.8)+(64516\times0.8)]$

$\therefore \tau_1 + 3.397(\tau_2+\tau_3) = 3718$ (2)

Now, considering the rotations of the cells and S_1, S_2, S_3, S_4, S_5 *and* S_6 as the length of cell walls,

We have,

$$\tau_1S_1 + \tau_4S_4 = 2G\theta A_1$$
$$-\tau_4S_4 + 2\tau_2S_2 + \tau_5S_5 = 2G\theta A_2 \quad (3)$$
$$-\tau_5S_5 + 2\tau_3S_3 + \tau_6S_6 = 2G\theta A_3$$

Here $S_1 = (\pi\times127) = 398mm$

$S_2 = S_3 = S_4 = S_5 = S_6 = 254mm$

$\therefore$ (3) can be written as

$$398\tau_1 + 254S_4 = 25322G\theta$$
$$-254\tau_2 + 2\times254\times\tau_2 + 254\tau_5 = 64516G\theta \quad (4)$$
$$-254\tau_2 + 2\times254\times\tau_3 + 254\tau_6 = 64516G\theta$$

Now, Solving (1), (2) and (4) we get

$\tau_1 = 40.4N/mm^2$

$\tau_2 = 55.2N/mm^2$

$\tau_3 = 48.9N/mm^2$

$\tau_4 = -12.7N/mm^2$

$\tau_6 = 36.6N/mm^2$