

6.1 Energy and Momentum Coefficients

Generally, in the energy and momentum equations the velocity is assumed to be steady uniform and non-varying vertically.

This assumption does not introduce any appreciable error in case of steady (or nearly uniform) flows. However, the boundary resistance modifies the velocity distribution. The velocity at the boundaries is less than the velocity at a distance from the boundaries.

Further, in cases where the velocity distribution is distorted such as in flow through sudden expansions/contractions or through natural channels or varying cross sections, error is introduced.

When the velocity varies across the section, the true mean velocity head across the section, $\left(v^2/2g\right)_m$, (the subscript m indicating the mean value) need not necessarily be equal to $\bar{v}^2/2g$. Hence, a correction factor is required to be used for both in energy and momentum equations (See Box). The mean velocity is usually calculated using continuity equation.

Keulegan presented a complete theoretical derivation of energy coefficient α and proved that the selection of α and β (Momentum coefficient) depends solely on the concept of the coefficient of friction which is adopted. If the equation of motion is derived by the energy method, the concept underlying the friction coefficient in that equation is that of energy dissipation in the fluid per unit length of channel and α is the proper factor to use. To understand proper use of factors α and β and the energy principle or momentum principle is used appropriately.

Box:

The weight of flow through an element of area dA is equal to $\rho g v dA$; the kinetic energy per unit weight of this flow is $\bar{v}^2/2g$; The rate of transfer of kinetic energy through this element is equal to

$$\rho g v dA \cdot \frac{v^2}{2g} = \frac{\rho g v^3}{2g} dA \quad (1)$$

Hence, the kinetic energy transfer rate of the entire flow is equal to

$$\int_0^A \rho g \frac{v^3}{2g} dA \quad (2)$$

and the total weight rate of flow is equal to $\rho g v dA$

$$\rho g Q = \rho g \bar{V} A$$

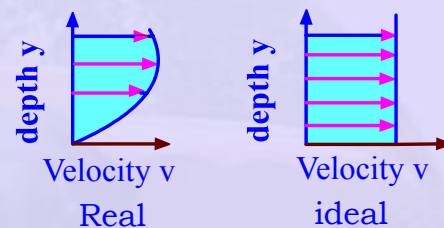
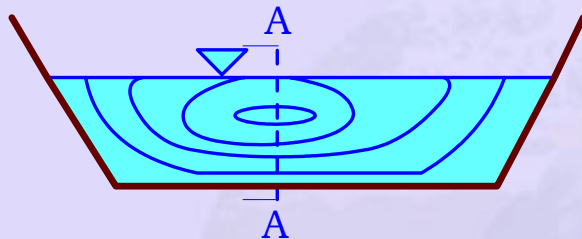
$$\text{mass density } \rho = \frac{\text{mass}}{\text{volume}}$$

$$\text{mass} = \rho * \text{volume} = \frac{\text{kg}}{\text{m}^3} \times \text{m}^3 = \text{kg}$$

$$\text{Force} = \text{N} = \text{mass} * \text{acceleration}$$

$$= \text{kg} * \frac{\text{m}}{\text{s}^2} = \frac{\text{kg m}}{\text{s}^2}$$

$$\text{specific weight } \gamma = \rho g = \frac{\text{kg}}{\text{m}^3} * \frac{\text{m}}{\text{s}^2} = \frac{\text{N}}{\text{m}^3}$$



Velocity distribution along section AA

Velocity distribution in a Trapezoidal Section

The mean velocity is by definition equal to Q / A . Hence, the mean velocity head, or kinetic energy per unit weight of fluid, is equal to

$$\left(\frac{v^2}{2g} \right)_m = \frac{\int_0^A \frac{v^3}{2g} dA}{\bar{V} A} = \alpha \left(\frac{\bar{V}^2}{2g} \right) \quad (3)$$

in which α is a correction coefficient to be applied to the velocity head as calculated from the mean velocity. It is also known as the Coriolis coefficient. Hence

$$\alpha = \frac{\int_0^A v^3 dA}{\bar{V}^3 A} \approx \frac{\sum_{i=1}^N v_i^3 dA}{\bar{V}^3 A} \quad i = 1, \dots, N \quad (4)$$

Similar approach can be applied for computing the momentum term $\beta \bar{V} Q$. The rate of transfer of momentum through an element of area dA is equal to $\rho v^2 dA$; Following similar logic as above the momentum correction coefficient can be obtained as

$$\beta = \frac{\int_0^A v^2 dA}{\bar{V}^2 A} \approx \frac{\sum_{i=1}^N v_i^2 dA}{\bar{V}^2 A} \quad i = 1, 2, \dots, N \quad (5)$$

β is also known as Boussinesq coefficient.

In general, the coefficients are assumed to be unity for channels of regular geometrical cross sections and fairly straight uniform alignment, as the effect of non uniform velocity distribution on the computation of velocity head and momentum is small when compared to other uncertainties involved in the computations. Table shows the values of α and β for selected situations.

Table: Values of α and β for selected situations (after Chow, 1958)

Channel	α			β		
	Minimum	Maximum	Average	Minimum	Maximum	Average
Regular channels, flumes, spillways	1.10	1.20	1.15	1.03	1.07	1.05
Natural streams and torrents	1.15	1.50	1.30	1.05	1.17	1.10
River under ice cover	1.20	2.00	1.50	1.07	1.33	1.17
River valley, over flooded	1.50	2.00	1.75	1.17	1.33	1.25

The kinetic energy correction factor α and momentum correction factor β can be expressed as (see box).

$$\alpha = \frac{\int_0^A v^3 dA}{\bar{V}^3 A} \approx \frac{\sum_{i=1}^N v_i^3 dA}{\bar{V}^3 A} \quad i = 1, \dots, N \quad (4)$$

$$\beta = \frac{\int_0^A v^2 dA}{\bar{V}^2 A} \approx \frac{\sum_{i=1}^N v_i^2 dA}{\bar{V}^2 A} \quad i = 1, 2, \dots, N \quad (5)$$

6.1.1 Determination of α and β

Many investigators have done extensive investigations on the computation of α and β .

Chow (1958) has summarised different equations for determination of α and β for various velocity distributions.

Rehbock assumed a linear velocity distribution and obtained

$$\alpha = 1 + \varepsilon^2$$

$$\beta = 1 + \frac{\varepsilon^2}{3}$$

and for logarithmic velocity distribution.

$$\alpha = 1 + 3\varepsilon^2 - 2\varepsilon^3$$

$$\beta = 1 + \varepsilon^2$$

in which $\varepsilon = \left\{ \frac{V_{\max}}{\bar{V}} - 1 \right\}$, V_{\max} is the maximum velocity and \bar{V} is the mean velocity

If the velocity distribution is along a vertical is logarithmic, then the relation between α and β , as shown by Bakhmateff, is that β exceeds unity by about one-third of the

amount by which α exceeds unity. If $\beta \approx 1 + n$ and $\alpha \approx 1 + 3n$ then $\beta = \frac{\alpha + 2}{3}$

approximately. Generally, the coefficients α and β are greater than one. They are both equal to unity when the flow is uniform across the section, and the farther, the flow departs from uniform, the greater the coefficients become. The form of Equations (4) and (5) makes it clear that α is more sensitive to velocity variation than β , so that for a given channel section, $\alpha > \beta$. Values of α and β can easily be calculated for idealized two-dimensional velocity distributions.

Velocity Distribution $\frac{v}{v_0} = \left(\frac{y}{y_0}\right)^n$
$\bar{v} = \frac{v_0}{n+1}$
$\alpha = \frac{(n+1)^3}{3n+1}$
$\beta = \frac{(n+1)^2}{2n+1}$
$\frac{\alpha-1}{\beta-1} = \frac{(n+3)(2n+1)}{(3n+1)}$
If $n = \frac{1}{7}$ $\alpha = 1.043, \beta = 1.015$

The high value of α appropriate to laminar flow is of limited interest, since laminar flow is rare in free surface flow problems. For turbulent flow in regular channels α seldom exceeds 1.15. In view of the limited experimental data on values of α , the question always arises whether the accuracy attainable with channel computations warrants its inclusion!

A practical method of arriving at the values of α and β for other than an idealised velocity distribution is a semi graphical and arithmetical solution based on planimeted areas of isovels plotted from data measurable at the cross section. Measured velocities are plotted to draw the Isovells. The Isovells are constructed for each cross section and cross sectional areas, ΔA , of each stream tube are calculated with planimeter and computations performed.

6.1.2 The Methods of computation of α and β may be classified as

1. Theoretical Methods

Based on experimental studies Strauss in 1967, has given empirical formulae for computing α and β for general channel section based on the velocity distribution given by the following equation.

$$V=ay^{1/n}$$

in which v is the velocity at a point located at a height y from the bed a is a constant and n is an exponent such that $1 \leq n \leq \infty$.

α and β can easily be computed using following equations, if the velocity distribution is known.

$$\alpha = \frac{\int_0^A v^3 dA}{\bar{V}^3 A}$$

$$\beta = \frac{\int_0^A v^2 dA}{\bar{V}^2 A}$$

Strauss states that the general velocity distribution of the type given by above equation covers all possible distributions by suitably choosing the value of n . In the limiting case when $n \rightarrow \infty$ the velocity distribution tends to become rectangular. At the other extreme when $n=1$, the velocity distribution is linear for which case $\alpha = 2$ and $\beta = 1.33$.

Strauss showed that

$$\alpha = f(n, \epsilon_1, B_1, \gamma_1)$$

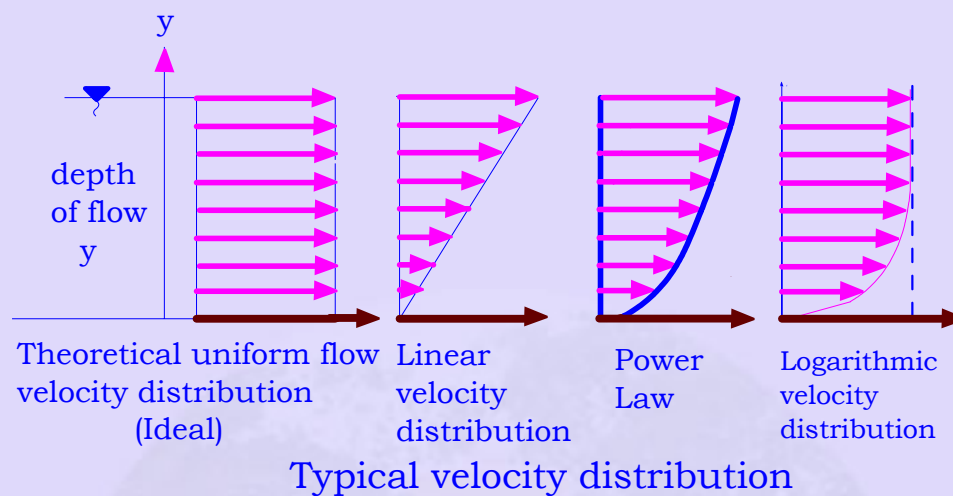
$$\beta = f(n, \epsilon_1, B_1, \gamma_1)$$

in which n is the exponent of the velocity distribution, and, ϵ_1 is normalized depths, B_1 is the normalized width of free surface to bed width, γ_1 is normalized bed width of berm (including) to channel bed. The velocity distribution plays a dominant role in influencing

α and β and in trapezoidal channel in addition to $B_1 = \frac{T}{b}$. For rectangular channel the

exponent n of velocity distribution has a dominating effect. But Strauss's method has limited practical utility. It is not always true that the same velocity distribution prevails along all the verticals of the cross-section, especially in non-rectangular channels. Also

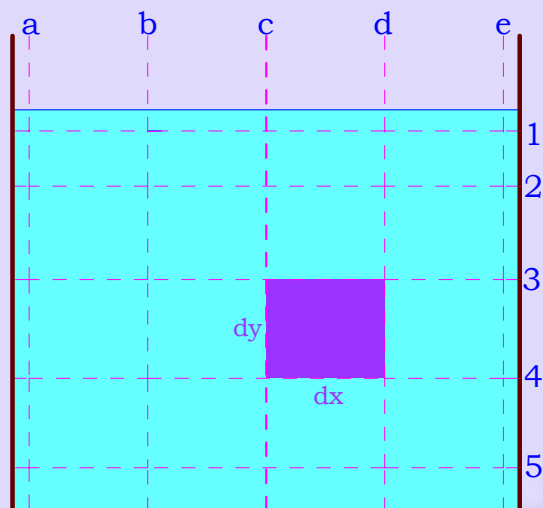
this method is not applicable when there is a negative velocity zone over the cross-section as in the case of a diverging channel, a bend or a natural channel.



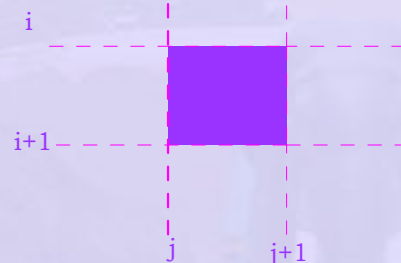
2. Graphical Method

In Velocity area method, the flow area is divided into number of grid cells and local velocities are measured using one of the measuring devices and finally integrating one will get the average velocity. The velocities are measured at the intersecting grid lines (nodes). Example: a1, b1, c1 etc.....a5, b5.....e5.

The average velocity over the elemental area is v_{cell} .



Grid for measuring Velocity



Co ordinates of the nodes are (i, j) , $(i+1, j)$, $(i+1, j+1)$, $(i, j+1)$

Corresponding velocities are $v(i, j)$, $v(i+1, j)$, $v(i+1, j+1)$, $v(i, j+1)$

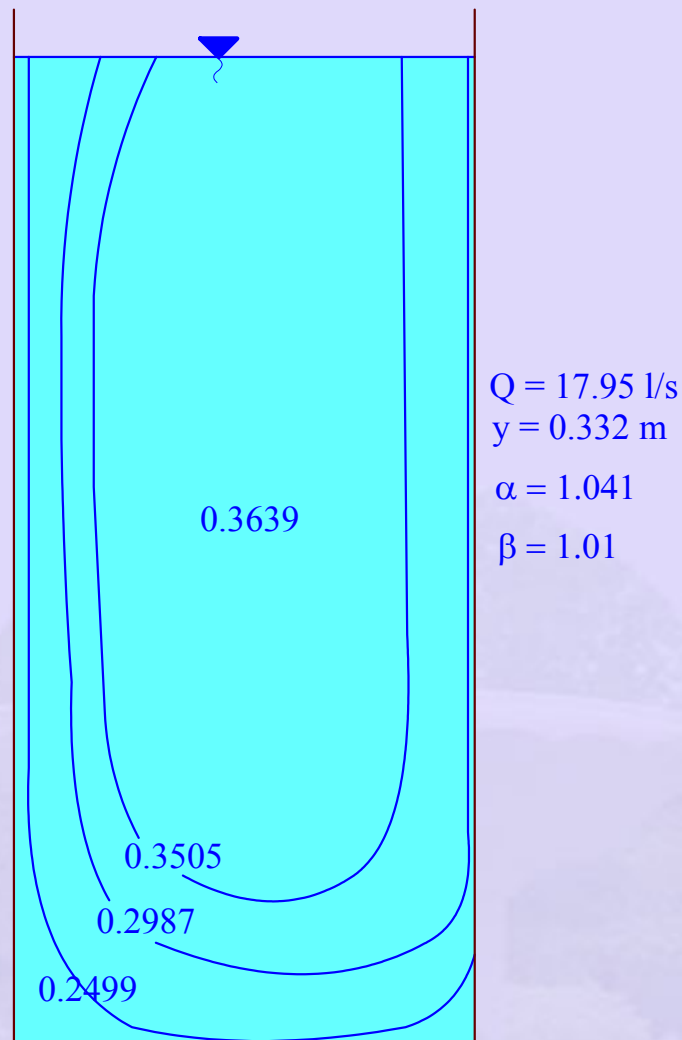
$$\text{Average velocity of the cell } \bar{v}_{\text{cell}} = \frac{v(i, j) + v(i+1, j) + v(i+1, j+1) + v(i, j+1)}{4}$$

Average velocity of the flow

$$\bar{v} = \frac{1}{A} \int_0^b \int_0^y v \, dy \, db \approx \frac{\sum \bar{v}_{\text{cell}} \, dA}{A (= by)}$$

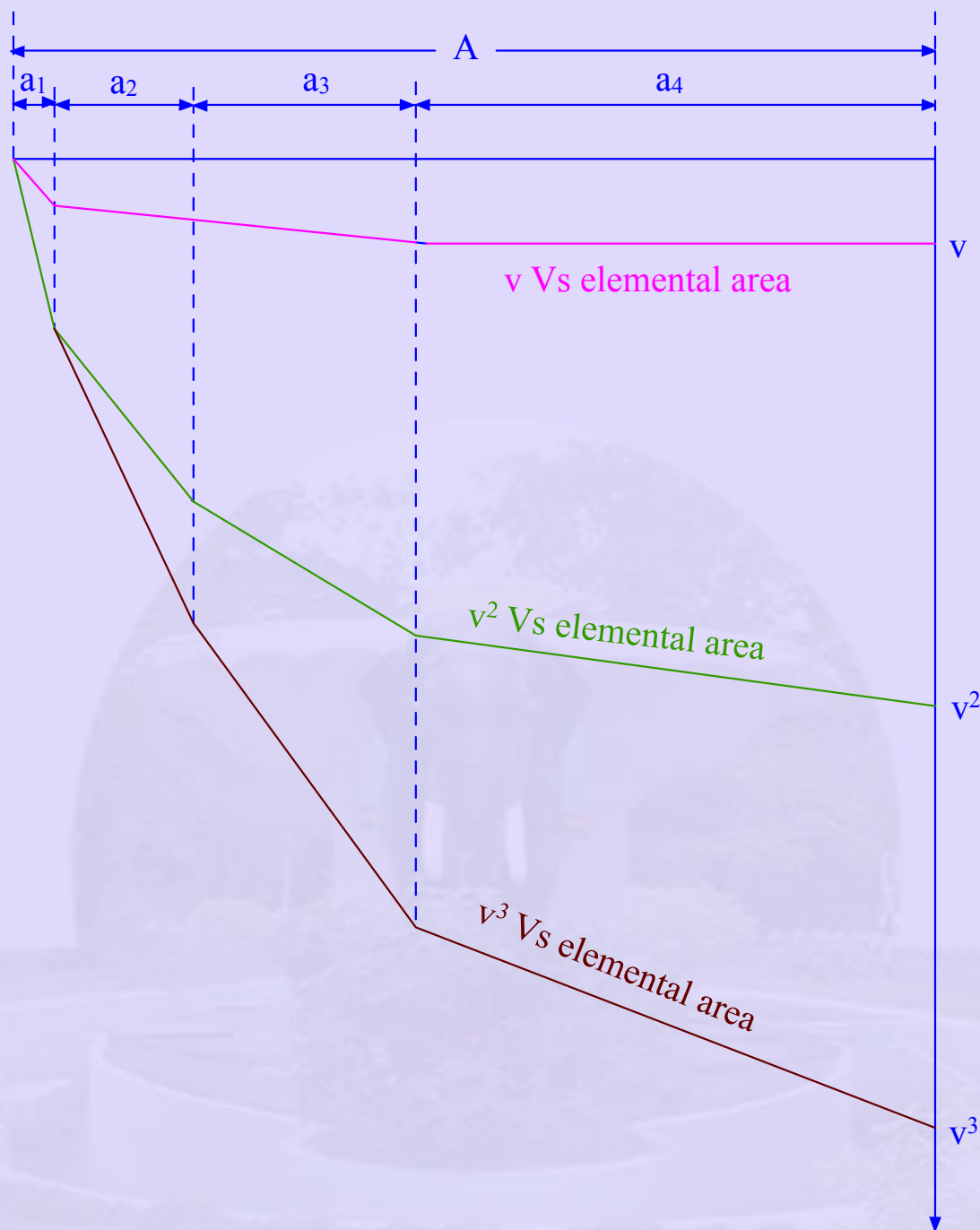
in which dA is the elemental area of the cell

The other alternative is to draw the isovels (isovel is a line having the same value of velocity sometimes it is also known as isopleths) assuming the linear variation between two values and interpolating the value in between two nodes. It may be noted that the velocity would be zero on the solid boundary. Hence the gradients are sharper very close to the boundary. Typical isovels are shown in Figure. In this method, velocities are measured at several points of cross-section and the lines of equal velocities called 'isovels' (also called isotachs) are drawn as shown in Figure.



Graphical Method

While drawing 'isovels' it is assumed that the velocity varies linearly between two points. Next the area within each isovel is plain metered. Assuming that the velocity through the area bounded by, two 'isovels' is equal to the average of their values α and β and are calculated using the following expressions.

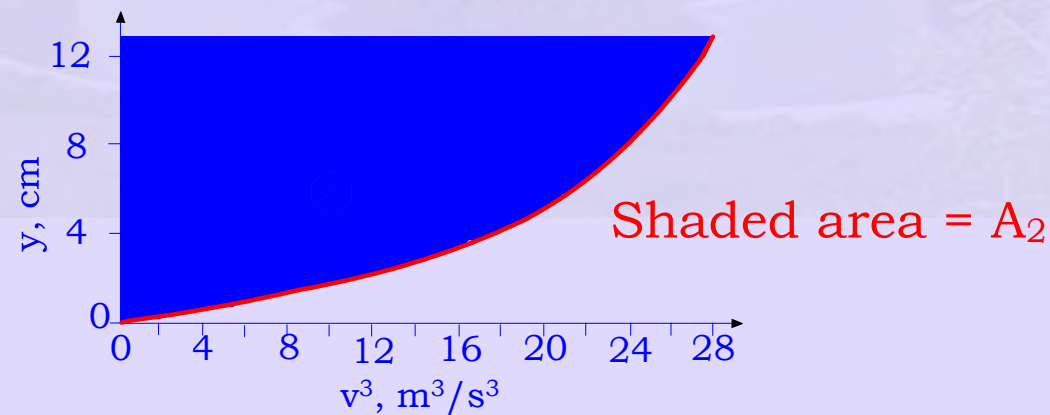
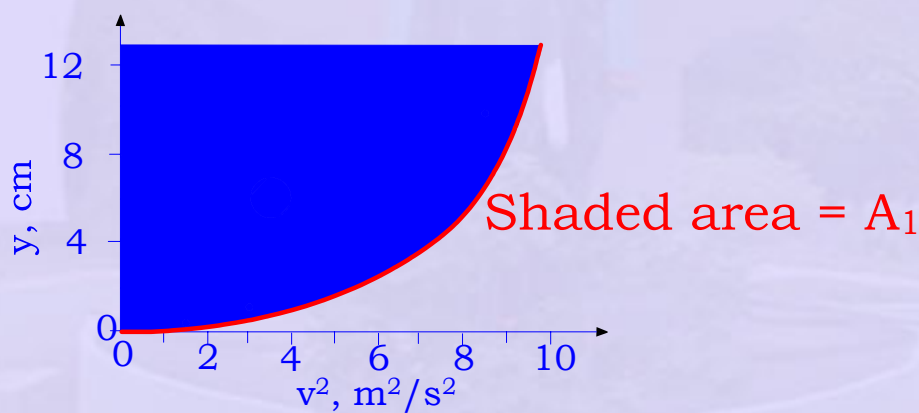
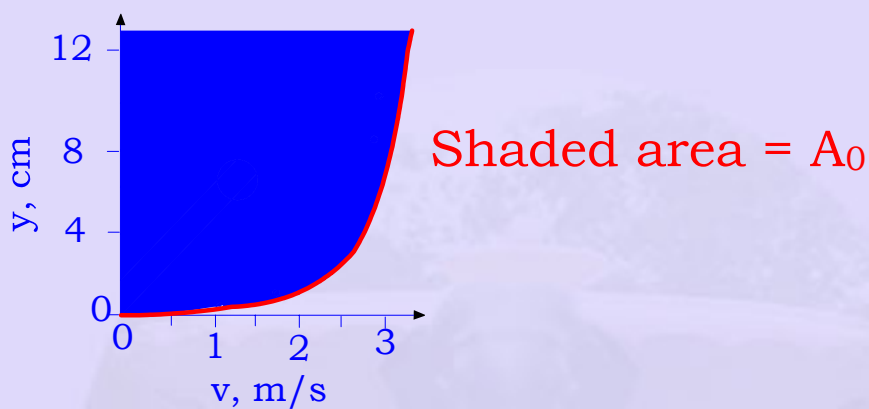


Graphical Method of determining α and β ($\sum av$, $\sum av^2$, $\sum av^3$)

$$\alpha = \frac{\int v^3 dA}{A\bar{V}^3} \approx \frac{\sum v^3 dA}{A\bar{V}^3} \quad (4) \text{ and}$$

$$\beta = \frac{\int v^2 dA}{A\bar{V}^2} \approx \frac{\sum v^2 dA}{A\bar{V}^2} \quad (5)$$

Rehbock used a graphical method and reduced the computational work in the above procedure. After planimetering the areas within each isovel, he plotted the curves of v , v^2 , and v^3 against the corresponding planimetered areas as shown in Figure. It is evident that the areas under v^2 , and v^3 curves are equal to $\sum v^3 dA$ and $\sum v^2 dA$ respectively. \bar{V} , α and β are computed as shown in the box.



Graphical method of computing \bar{V} , α and β

Shaded areas A_0 , A_1 , A_2 are planimetered.

The average velocity

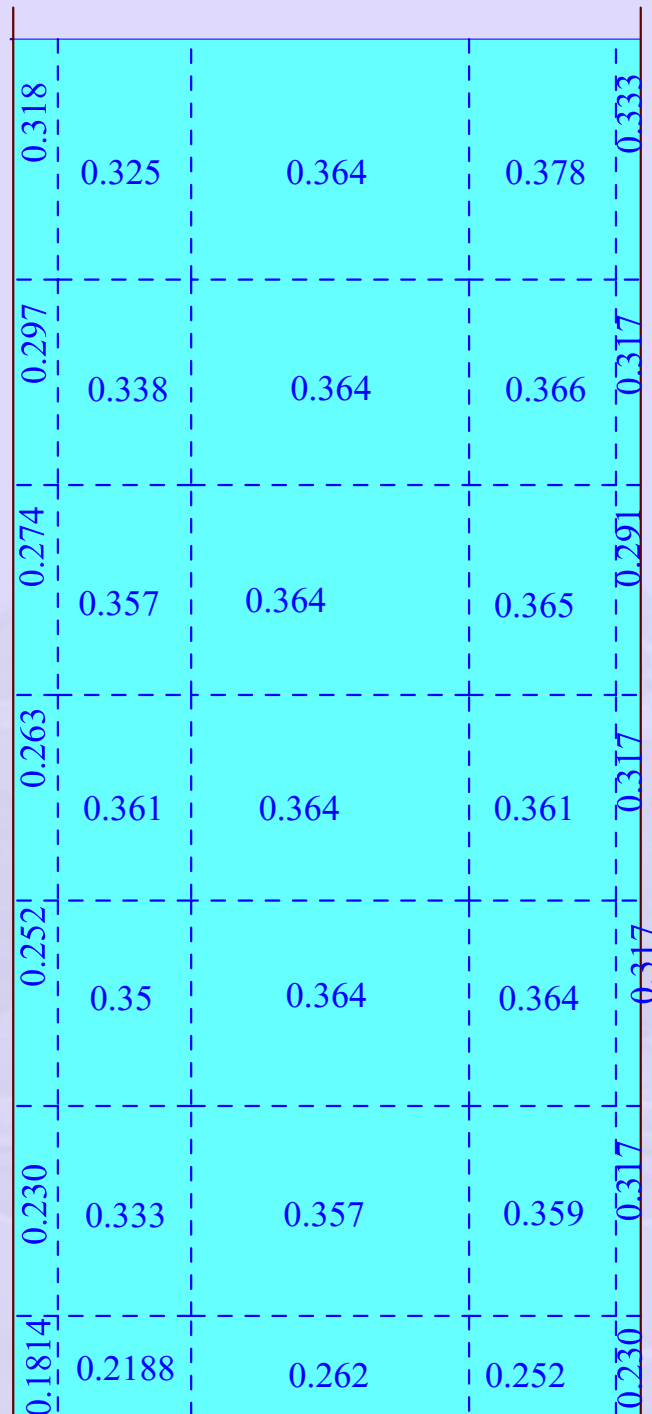
$$\bar{V} = \int_1^y v \, dy = \frac{\text{shaded area } A_0}{y}$$

$$\text{Similarly, } \beta = \frac{\text{shaded area } A_1}{\bar{V}^2 y}$$

$$\text{and } \alpha = \frac{\text{shaded area } A_2}{\bar{V}^3 y}$$

6.1.3 Grid Method

In this method, the flow area is divided into suitably chosen grids and velocities at the centers of gravity of these grids are measured as shown in Figure 3. Assuming that the effective velocity through each grid is equal to that at the center of gravity of the grid, the quantities $\sum v \, da$, $\sum v^2 da$, $\sum v^3 da$ are computed. In particular if the grids are so chosen that their areas are equal, the computational work becomes relatively easier. However, for greater accuracy the size of the grid should be chosen as small as possible. Also near the boundaries, relatively smaller grids are to be chosen. The advantage of this method is that it is less time-consuming than the graphical method as the actual velocities need not be calculated and isovels need not be drawn. For purposes of comparison, α and β for rectangular channel shown in the above figure are computed by this method and are given in the following figure.



$$Q = 17.95 \text{ l/s}$$

$$y = 0.332 \text{ m}$$

$$\alpha = 1.041$$

$$\beta = 1.024$$

Grid Method

6.1.4 Methods based on the use of empirical formula

Assuming a linear velocity distribution law Rehbock has proposed the following formulae for approximate values of α and β .

$$\alpha = 1 + \epsilon^2; \quad \beta = 1 + \frac{\epsilon^2}{3}$$

In which $\epsilon = \frac{V_{\max}}{\bar{V}} - 1$

Assuming a logarithmic velocity distribution law proposed the following expressions.

$$\alpha = 1 + 3\epsilon^2 - 2\epsilon^3; \quad \beta = 1 + \epsilon^2$$

In which v_{\max} is the maximum velocity and \bar{V} is the mean velocity.

It should be noted that the above approximate formulae are applicable only when the flow is free from any reverse flow occurring over any part of the cross-section of flow.

6.1.5 Computation of α and β for Reverse Flow

In case of the reverse flow one of the four methods presented above is directly applicable. If the reverse flow is occurring over any part of the cross-section of the flow, α and β can be calculated using either the graphical or the grid method. While using these methods it should be noted that the velocity in the reverse flow region should be assigned a negative sign and all the computations should be done taking the sign also into consideration.

6.1.6 Values of α and β in Several Practical Cases

Actual values α and β in many practical cases (which are frequently met with in Hydraulic Engineering) are presented in Table I. Some of these values are listed by O'Brien and Hickox O'Brien and Johnson and King. They are reproduced here along with several other cases for the sake of a comprehensive table of α and β values.

Sl. No.	Channel Dimensions				Hydraulic elements		Coefficients			Remarks
	width (m)	Max. depth (m)	Hydraulic Radius (m)	Area (m ²)	Critical depth (m)	Mean velocity (m/s)	Graphical	Rehbock	Graphical	
1	0.60	0.862	0.222	0.519	0.198	0.320	1.20	1.10	1.07	Rectangular channel 0.9144 m above weir and obstructions upstream
2	1.00	0.862	0.3250	0.895	0.216	0.53	1.22	1.20	1.08	
3	1.00	0.874	0.3249	0.893	0.219	0.365	1.41	1.37	1.12	
4	1.01	0.429	0.2316	0.431	0.496	2.56	1.07	1.04	1.03	Simson Tunnel - centre of straight reach 49.98 m long
5	10.54	3.23	1.86	23.27	1.42	1.01	1.10	1.07	1.05	
6	1.987	1.50	0.6309	2.898	0.76	1.48	1.07	1.03	1.034	Horse shoe conduit straight reach
7	159.4	3.81	2.438	4.055	1.91	1.024	1.35	1.43	1.121	Rhine 365.76 m below bridge on a long curve
8	2.59	1.38	0.6949	3.429	0.685	0.886	1.06	1.02	1.01	Sudbury Aqueduct with a bottom slope 0.000189
9	2.67	1.22	0.6492	3.009	0.658	0.874	1.04	1.04	1.014	
10	2.74	0.914	0.548	2.19	0.600	0.792	1.04	1.03	1.014	
11	2.71	0.618	0.411	1.415	0.53	0.658	1.04	1.02	1.010	
12	2.65	0.460	0.326	1.014	0.499	0.569	1.04	1.03	1.012	
13		0.264		0.053	0.35	2.31	1.161			Computed with Bazin series 10
14		0.244		0.0366	0.14	0.205	1.138			Computed from Nikurade's data
15	1.286	0.762					1.07			Series (E) Schoder and Turner
16	1.286	1.524					1.08			Series (I) Schoder and Turner - Run 54 to 58.
17	1.286	1.524					1.60			Series I ibid
18	1.286	1.524					2.08			Series I ibid
19	1.286	3.07					1.80			Series D Schoder and Turner Runs 101 to 105
20	1.286	2.743					2.00			Series D, L, M. Schoder and Turner
21							1.528	1.105		Triangular channel
22							1.665	1.225		Trapezoidal channel
23							1.365	1.085		Pipe
24							1.460	1.164		Shallow ditch
25							1.422	1.136		Natural channel
26	0.45	0.0911						1.222		Experiment number 2C - Rajaratnam Muralidhar

27	0.45	0.350					3.72	2.14		Diverging channel
28							1.76	1.41		Rectangular open channel bend
29	0.61						15.40	5.00		Maximum α and β in a hydraulic jump with an inflow Froude number of 7.4.
30							3.87			At the outlet section of a draft tube
31							7.40			Spiral flow under a model turbine wheel

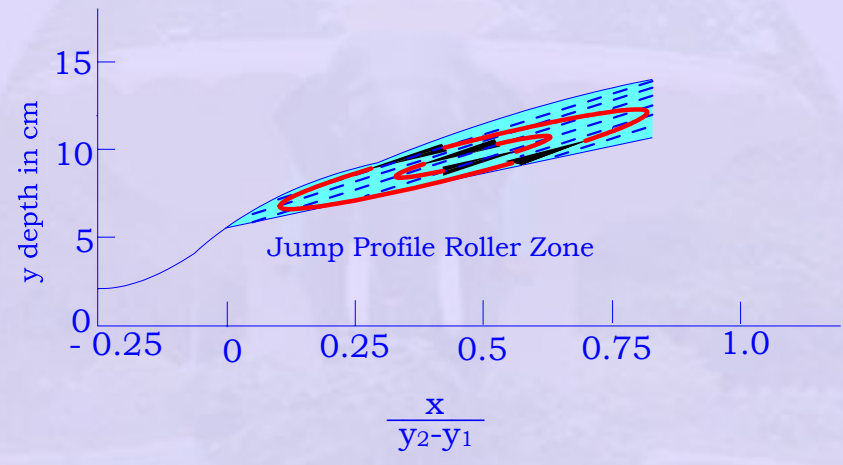
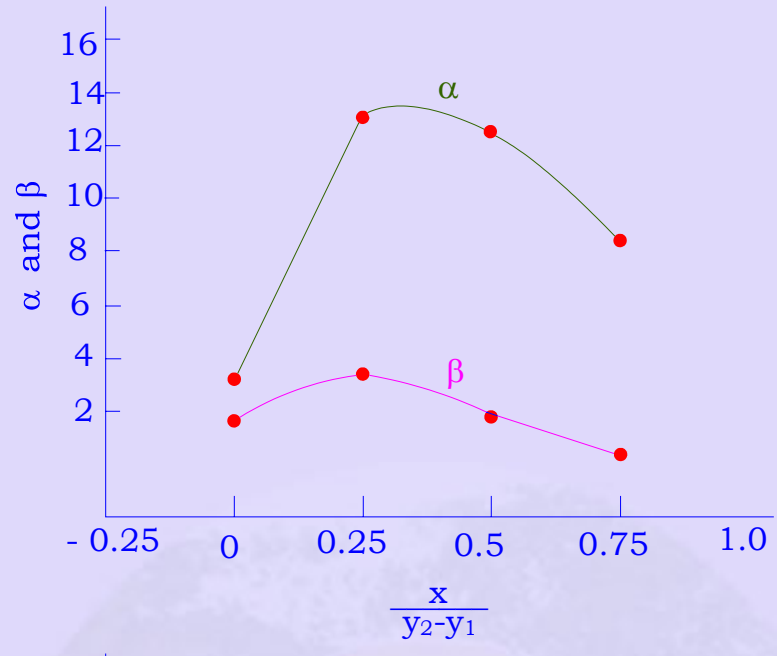
(Serial No. 1 to 20 are from O'Brien and Johnson, Enr, Vol. 1113, page 214 - 216, 1934 August 16 th after Jagannadhar Rao and others).

From the table it may be seen that α values are larger in non-rectangular channels compared to rectangular channels and also that the values for natural channels are as high as 1.422. When there is a reverse flow in the cross-section, the values of α are still larger. The value in the case of a diverging channel is 3.72. For spiral flows a value of α as high as 7.4 has been quoted . All these examples show that there are several practical cases in which the neglect of α and β in hydraulic flow computations for a proper assessment of energy and momentum at any flow section may lead to large errors.

6.1.7 Variation of α and β along the Hydraulic jump

The variation of α and β along the length of hydraulic jump is given in figure below.

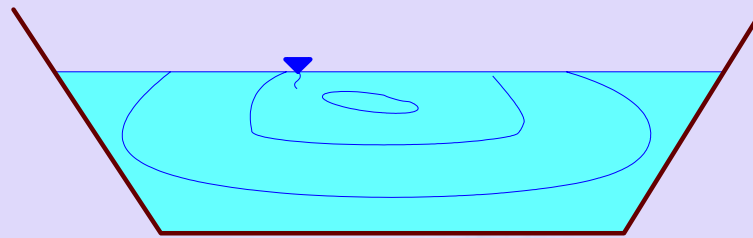
Jagannadha Rao (1970) conducted the experiments in a flume of 0.6 m width at Indian Institute of Technology, Kharagpur. The data given is for the case of a hydraulic jump with an approach flow Froude number of 7.4.



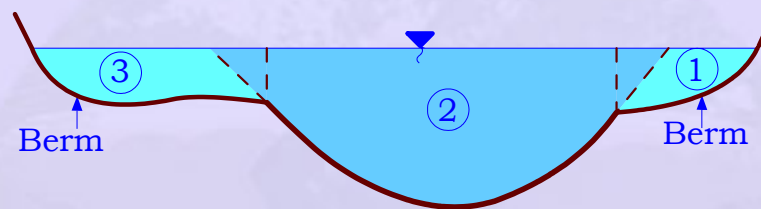
Variation of α and β along the hydraulic jump

6.1.8 α, β for Flow in Natural Channels

The natural channels can be subdivided into distinct regions, each with a different mean velocity.



Isovels in a single channel α is nearly 1.15



② Main channel (MC)

① and ③ channel in the flood plains

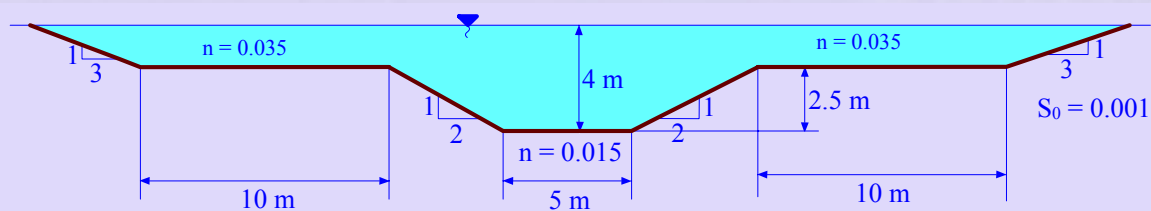
natural channel: River $\alpha \approx 2.0$

Typical Cross Sections of natural channel

$$\alpha = \frac{v_1^3 A_1 + v_2^3 A_2 + v_3^3 A_3}{\bar{V}^3 (A_1 + A_2 + A_3)}$$

$$\beta = \frac{v_1^2 A_1 + v_2^2 A_2 + v_3^2 A_3}{\bar{V}^2 (A_1 + A_2 + A_3)}$$

$$\bar{V} = \frac{v_1 A_1 + v_2 A_2 + v_3 A_3}{(A_1 + A_2 + A_3)}$$



COMPOUND CANAL CROSS-SECTION

This is particularly true in time of flood, when the river overflows on to its flood plains, or "berms,". These are known as Compound channel. In this case there are in effect three

separate channels. The mean velocity over the berms will be less than that in the main channel (MC), because of higher resistance to flow (basically due to, smaller depths over the berms , and due to the higher roughness in the berms. This variation in mean velocity among the different flow zones (Main channel and berms) is mainly responsible for values of α much higher than those produced by gradual variation within a given section, so much higher as virtually to nullify any contribution to the value of α produced by gradual velocity variation. However, it is usually accurate enough to compute by assuming the velocity to be constant within each subsection (zone) of the waterway; then the following may be written.

$$\alpha = \frac{v_1^3 A_1 + v_2^3 A_2 + v_3^3 A_3}{\bar{V}^3 (A_1 + A_2 + A_3)}$$

$$\alpha = \frac{v_1^3 A_1 + v_2^3 A_2 + v_3^3 A_3}{\left(\frac{v_1 A_1 + v_2 A_2 + v_3 A_3}{A_1 + A_2 + A_3} \right) (A_1 + A_2 + A_3)}$$

$$\alpha = \frac{(v_1^3 A_1 + v_2^3 A_2 + v_3^3 A_3) (A_1 + A_2 + A_3)^2}{(v_1 A_1 + v_2 A_2 + v_3 A_3)^3}$$

$$= \frac{\left(\sum_{i=1}^N v_i^3 A_i \right) \left(\sum_{i=1}^N A_i^2 \right)}{\sum_{i=1}^N (v_i A_i)^3}$$

Similarly expression for β can be obtained.

$$\beta = \frac{v_1^2 A_1 + v_2^2 A_2 + v_3^2 A_3}{\bar{V}^2 (A_1 + A_2 + A_3)}$$

in which $\bar{V} = \frac{v_1 A_1 + v_2 A_2 + v_3 A_3}{A_1 + A_2 + A_3}$.

When flow resistance formula (Manning, Chezy, other formulae) is combined with the above equations numerical values of α , may exceed much higher than 2 under certain situations. Generally, the α value is taken as 1.0 when the information is lacking.

References:

1. Chow Van Te Open Channel Hydraulics, McGraw Hill Publications, 1958.
2. Henderson F.M. Open Channel Flow, MacMillan Publishing Company, 1966.
3. Jaganadha Rao, M.V., Lakshmana Rao, N.S., and Seetharamiah, K. "On the use of Energy and Momentum coefficients in Hydraulic flow computations" - Journal - Irrigation Power CBIP , Volume 27, part- 3, pp 315 - 326, 1970.
4. Strauss.V. "The Kinetic Energy Correction Factor and the Momentum Correction Factor in Open Channels". Proceedings of Twelfth Congress of I.A.H.R., Vol.1, Sept 1967,pp.314-323.
5. O' Brien, M.P. : "Discussion on stream flow in general terms" by Casler", Trans. A.S.C.E. Vol. 94, 1930, pp. 42 - 47.
6. O' Brien, M.P. and Johnson, J. W. : "Velocity Head Connections for Hydraulic Flows". Engineering News Record. Vol. 113, No. 7, pp. 214 - 216, Aug. 16, 1934.

