Module 4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 30

The Direct Stiffness Method: Plane Frames

Instructional Objectives

After reading this chapter the student will be able to
1. Derive plane frame member stiffness matrix in local co-ordinate system.
2. Transform plane frame member stiffness matrix from local to global co-ordinate system.
3. Assemble member stiffness matrices to obtain the global stiffness matrix of the plane frame.
4. Write the global load-displacement relation for the plane frame.
5. Impose boundary conditions on the load-displacement relation.
6. Analyse plane frames by the direct stiffness matrix method.

30.1 Introduction

In the case of plane frame, all the members lie in the same plane and are interconnected by rigid joints. The internal stress resultants at a cross-section of a plane frame member consist of bending moment, shear force and an axial force. The significant deformations in the plane frame are only flexural and axial. In this lesson, the analysis of plane frame by direct stiffness matrix method is discussed. Initially, the stiffness matrix of the plane frame member is derived in its local co-ordinate axes and then it is transformed to global co-ordinate system. In the case of plane frames, members are oriented in different directions and hence before forming the global stiffness matrix it is necessary to refer all the member stiffness matrices to the same set of axes. This is achieved by transformation of forces and displacements to global co-ordinate system.

30.2 Member Stiffness Matrix

Consider a member of a plane frame as shown in Fig. 30.1a in the member co-ordinate system $x'y'z'$. The global orthogonal set of axes xyz is also shown in the figure. The frame lies in the xy plane. The member is assumed to have uniform flexural rigidity EI and uniform axial rigidity EA for sake of simplicity. The axial deformation of member will be considered in the analysis. The possible displacements at each node of the member are: translation in x'- and y'-direction and rotation about z'- axis.

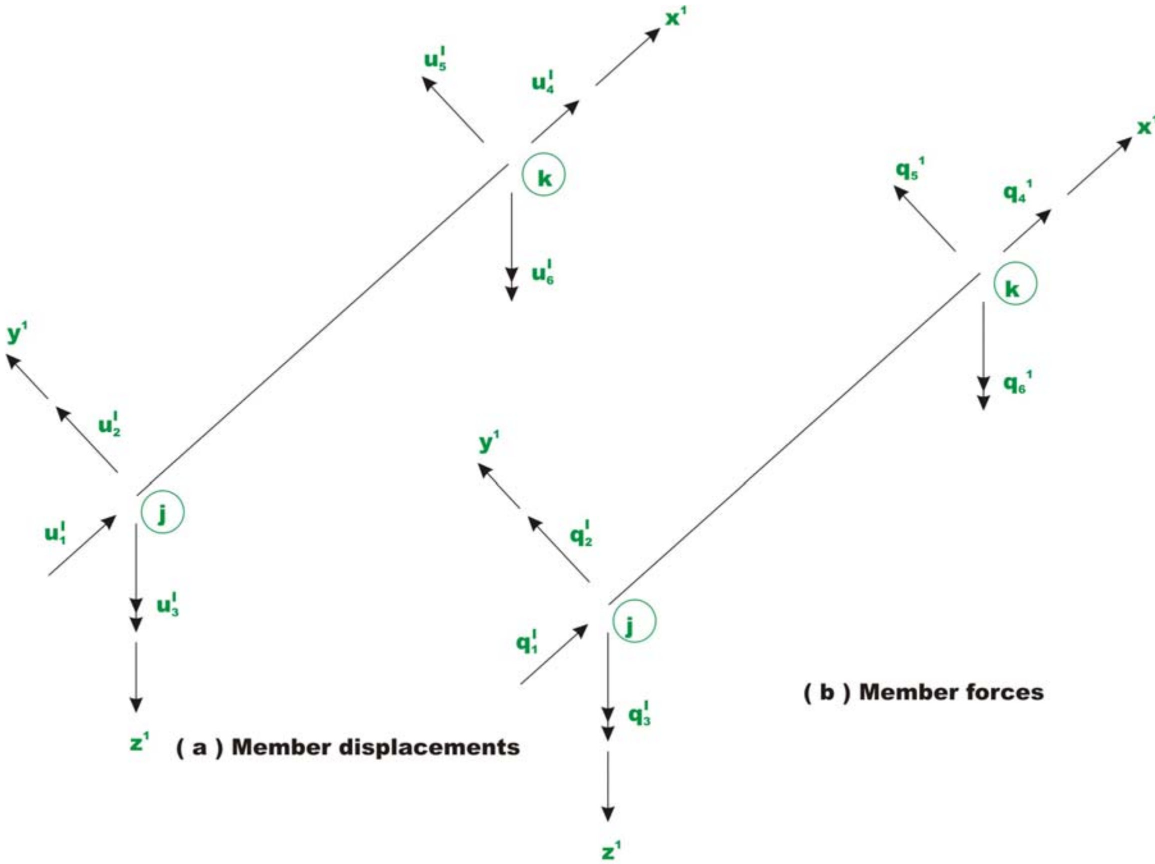

Fig. 30.1 Frame member in local co-ordinate system

Thus the frame members have six (6) degrees of freedom and are shown in Fig.30.1a. The forces acting on the member at end j and k are shown in Fig. 30.1b. The relation between axial displacement and axial forces is derived in chapter 24. Similarly the relation between shear force, bending moment with translation along y' axis and rotation about z' axis are given in lesson 27. Combining them, we could write the load-displacement relation in the local co-ordinate axes for the plane frame as shown in *Fig 30.1a, b* as,

$$
\begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \end{Bmatrix} =
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{bmatrix}
\begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{Bmatrix} \quad (30.1a)
$$

This may be succinctly written as

$$
\{q'\} = [k']\{u'\} \quad (30.1b)
$$

where $[k']$ is the member stiffness matrix. The member stiffness matrix can also be generated by giving unit displacement along each possible displacement degree of freedom one at a time and calculating resulting restraint actions.

30.3 Transformation from local to global co-ordinate system

30.3.1 *Displacement transformation matrix*

In plane frame the members are oriented in different directions and hence it is necessary to transform stiffness matrix of individual members from local to global co-ordinate system before formulating the global stiffness matrix by assembly. In Fig. 30.2a the plane frame member is shown in local coordinate axes $x'y'z'$ and in Fig. 30.2b, the plane frame is shown in global coordinate axes xyz. Two ends of the plane frame member are identified by j and k. Let u'_1, u'_2, u'_3 and u'_4, u'_5, u'_6 be respectively displacements of ends j and k of the member in local coordinate system $x'y'z'$. Similarly u_1, u_2, u_3 and u_4, u_5, u_6 respectively are displacements of ends j and k of the member in global co-ordinate system.

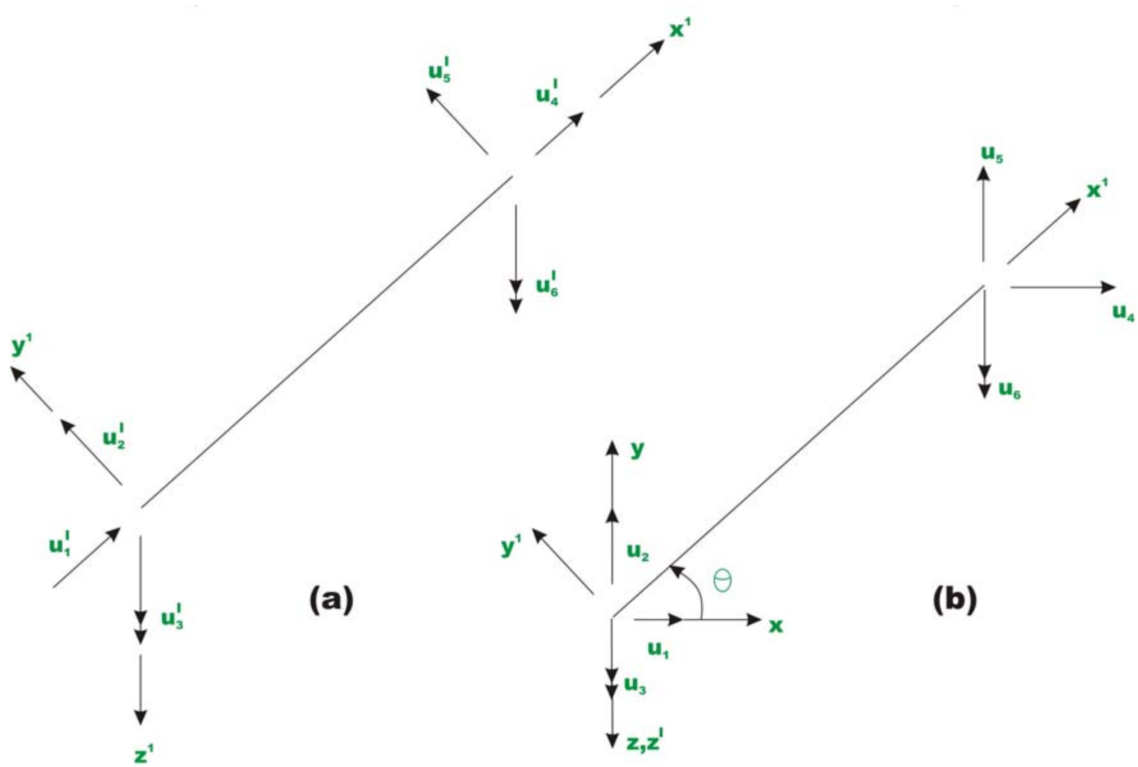

Fig. 30.2 Plane frame member in
(a) Local co-ordinate system
(b) Global co-ordinate system.

Let θ be the angle by which the member is inclined to global x-axis. From Fig.30.2a and b, one could relate u'_1, u'_2, u'_3 to u_1, u_2, u_3 as,

$$u'_1 = u_1 \cos\theta + u_2 \sin\theta \tag{30.2a}$$

$$u'_2 = -u_1 \sin\theta + u_2 \cos\theta \tag{30.2b}$$

$$u'_3 = u_3 \tag{30.2c}$$

This may be written as,

$$\begin{Bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \\ u'_6 \end{Bmatrix} = \left[\begin{array}{ccc:ccc} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \tag{30.3a}$$

Where, $l = \cos\theta$ and $m = \sin\theta$.

This may be written in compact form as,

$$\{u'\} = [T]\{u\} \tag{30.3b}$$

In the above equation, $[T]$ is defined as the displacement transformation matrix and it transforms the six global displacement components to six displacement components in local co-ordinate axes. Again, if the coordinate of node j is (x_1, y_1) and coordinate of node k are (x_2, y_2), then,

$$l = \cos\theta = \frac{x_2 - x_1}{L} \quad \text{and} \quad m = \sin\theta = \frac{y_2 - y_1}{L}.$$

Where $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (30.4)

30.3.2 *Force displacement matrix*

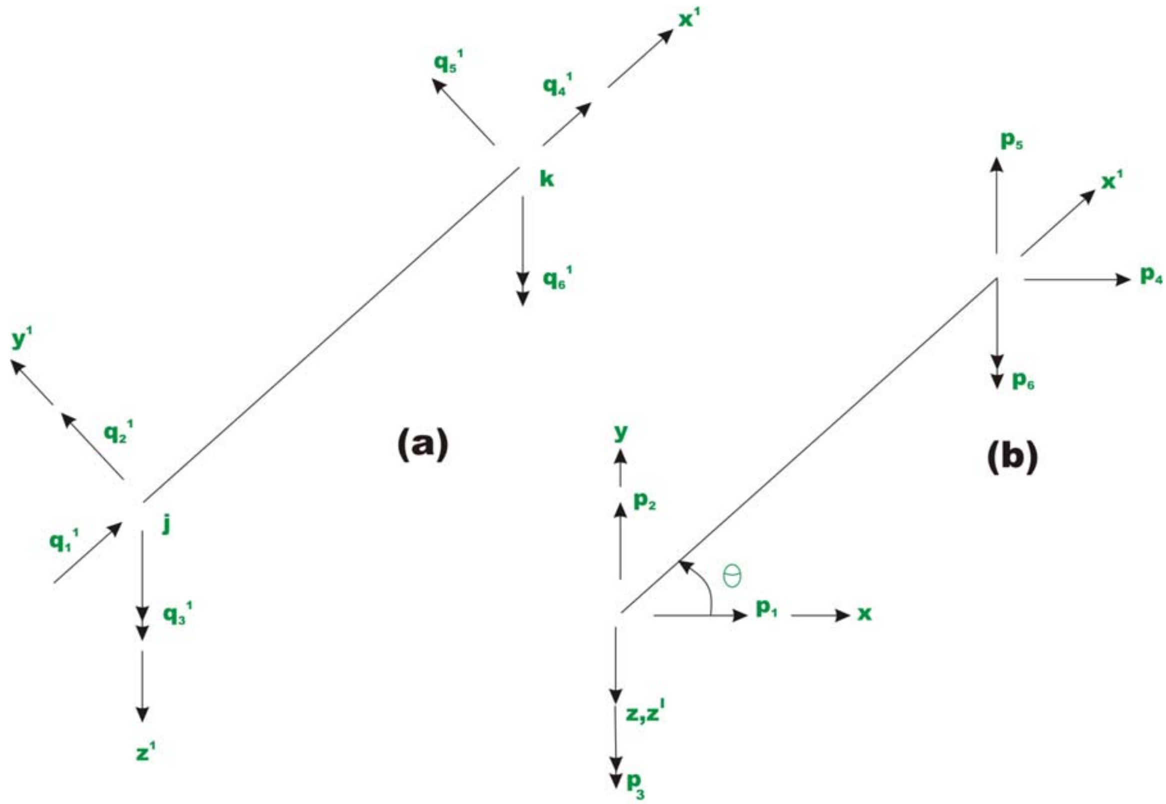

Fig. 30.3 Plane frame member in
(a) Local co-ordinate axes and
(b) In global co-ordinate system

Let q'_1, q'_2, q'_3 and q'_4, q'_5, q'_6 be respectively the forces in member at nodes j and k as shown in Fig. 30.3a in local coordinate system. p_1, p_2, p_3 and p_4, p_5, p_6 are the forces in members at node j and k respectively as shown in Fig. 30.3b in the global coordinate system. Now from Fig 30.3a and b,

$$p_1 = q'_1 \cos\theta - q'_2 \sin\theta \tag{30.5a}$$

$$p_2 = q'_1 \sin\theta + q'_2 \cos\theta \tag{30.5b}$$

$$p_3 = q'_3 \tag{30.5c}$$

Thus the forces in global coordinate system can be related to forces in local coordinate system by

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} l & -m & 0 & 0 & 0 & 0 \\ m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & l & -m & 0 \\ 0 & 0 & 0 & m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ q'_5 \\ q'_6 \end{Bmatrix} \tag{30.6a}$$

Where, $l = \cos\theta$ and $m = \sin\theta$.

This may be compactly written as,

$$\{p\} = [T]^T \{q'\} \tag{30.6b}$$

30.3.3 *Member global stiffness matrix*

From equation (30.1b), we have

$$\{q'\} = [k']\{u'\}$$

Substituting the above value of $\{q'\}$ in equation (30.6b) results in,

$$\{p\} = [T]^T [k']\{u'\} \tag{30.7}$$

Making use of equation (30.3b), the above equation may be written as

$$\{p\} = [T]^T [k'][T]\{u\} \tag{30.8}$$

or

$$\{p\} = [k]\{u\} \tag{30.9}$$

The equation (30.9) represents the member load-displacement relation in global coordinate system. The global member stiffness matrix $[k]$ is given by,

$$[k] = [T]^T [k'][T] \tag{30.10}$$

After transformation, the assembly of member stiffness matrices is carried out in a similar procedure as discussed for truss. Finally the global load-displacement equation is written as in the case of continuous beam. Few numerical problems are solved by direct stiffness method to illustrate the procedure discussed.

Example 30.1

Analyze the rigid frame shown in Fig 30.4a by direct stiffness matrix method. Assume $E = 200\,GPa$; $I_{ZZ} = 1.33 \times 10^{-4}\, m^4$ and $A = 0.04\, m^2$. The flexural rigidity EI and axial rigidity EA are the same for both the beams.

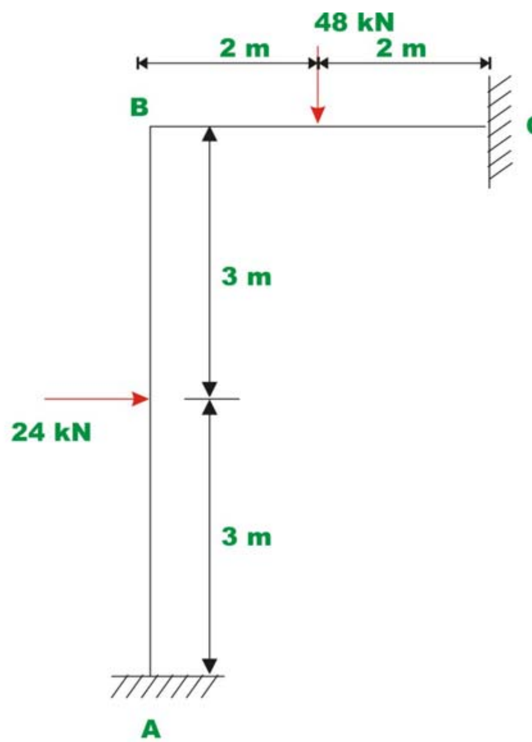

Fig. 30.4a Rigid Frame.

Solution:

The plane frame is divided in to two beam elements as shown in Fig. 30.4b. The numbering of joints and members are also shown in Fig. 30.3b. Each node has three degrees of freedom. Degrees of freedom at all nodes are also shown in the figure. Also the local degrees of freedom of beam element are shown in the figure as inset.

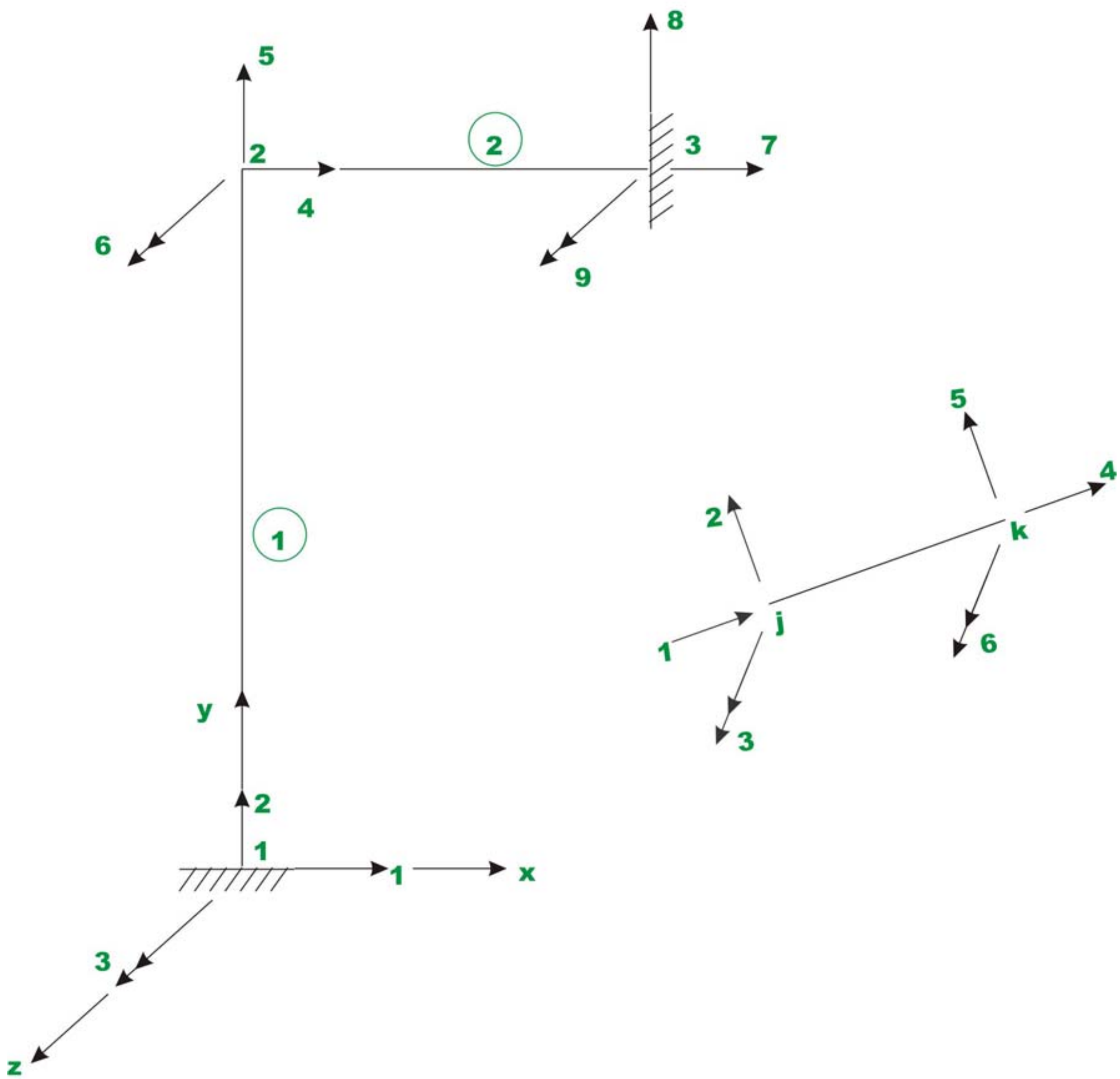

Fig. 30.4b Node and member numbering.

Formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. The origin of the global co-ordinate system is taken at node 1. Here the element stiffness matrix in global co-ordinates is only given.

Member 1: $L = 6\ m$; $\theta = 90°$ node points 1-2; $l = 0$ and $m = 1$.

$$[k^1] = [T]^T [k'] \ [T]$$

$$\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
[k^1] = & \begin{bmatrix}
1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\
0 & 1.333\times10^6 & 0 & 0 & -1.333\times10^6 & 0 \\
4.44\times10^3 & 0 & 17.78\times10^3 & 4.44\times10^3 & 0 & 8.88\times10^3 \\
1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\
0 & -1.333\times10^6 & 0 & 0 & 1.333\times10^6 & 0 \\
4.44\times10^3 & 0 & 8.88\times10^3 & 4.44\times10^3 & 0 & 17.78\times10^3
\end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}$$

(1)

Member 2: $L = 4\ m$; $\theta = 0°$; node points 2-3 ; $l = 1$ and $m = 0$.

$$[k^2] = [T]^T [k'][T]$$

$$\begin{array}{cc}
 & \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
= & \begin{bmatrix}
2.0\times10^6 & 0 & 0 & -2.0\times10^6 & 0 & 0 \\
0 & 5\times10^3 & 10\times10^3 & 0 & -5\times10^3 & 10\times10^3 \\
0 & 10\times10^3 & 26.66\times10^3 & 0 & -10\times10^3 & 8.88\times10^3 \\
-2.0\times10^6 & 0 & 0 & 2.0\times10^6 & 0 & 0 \\
0 & -5\times10^3 & -10\times10^3 & 0 & 5\times10^3 & -10\times10^3 \\
0 & 10\times10^3 & 8.88\times10^3 & 0 & -10\times10^3 & 26.66\times10^3
\end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\end{array}$$

(2)

The assembled global stiffness matrix $[K]$ is of the order 9×9. Carrying out assembly in the usual manner, we get,

$$[K]=\left[\begin{array}{ccc|ccc|ccc}
1.48 & 0 & -4.44 & -1.48 & 0 & -4.44 & 0 & 0 & 0 \\
0 & 1333.3 & 0 & 0 & -1333.3 & 0 & 0 & 0 & 0 \\
-4.44 & 0 & 17.77 & 4.44 & 0 & 8.88 & 0 & 0 & 0 \\
\hline
-1.48 & 0 & 4.44 & 2001.5 & 0 & 4.44 & -2000 & 0 & 0 \\
0 & -1333.3 & 0 & 0 & 1338.3 & 10 & 0 & -5 & 10 \\
-4.44 & 0 & 8.88 & 4.44 & 10 & 44.44 & 0 & -10 & 13.33 \\
\hline
0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\
0 & 0 & 0 & 0 & 10 & 13.33 & 0 & -10 & 26.66
\end{array}\right] \quad (3)$$

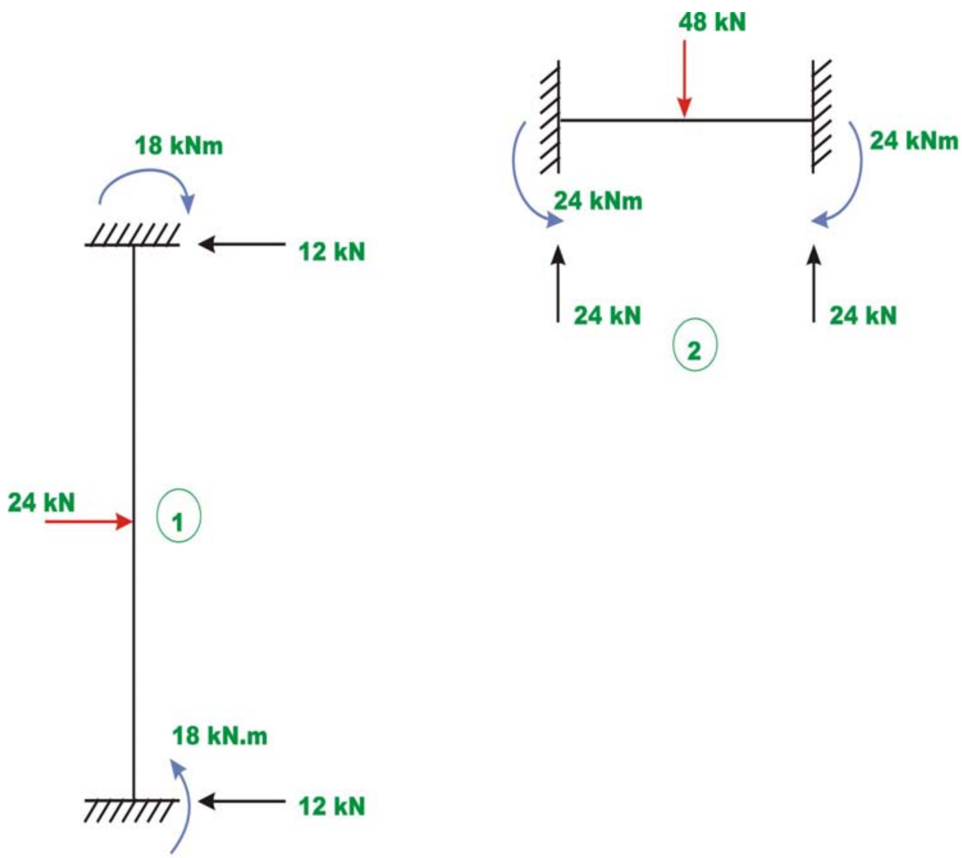

Fig. 30.4c Fixed end action due to external load in element (1) and (2)

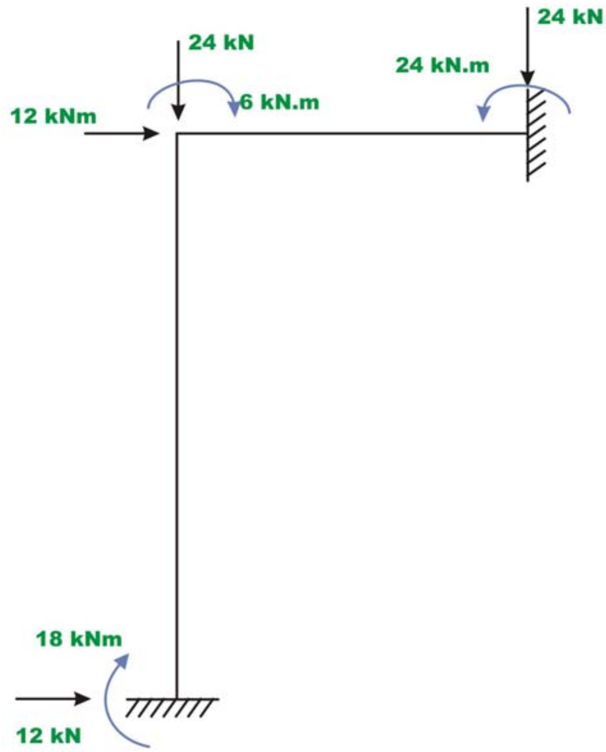

Fig. 30.4d Equivalent joint loads.

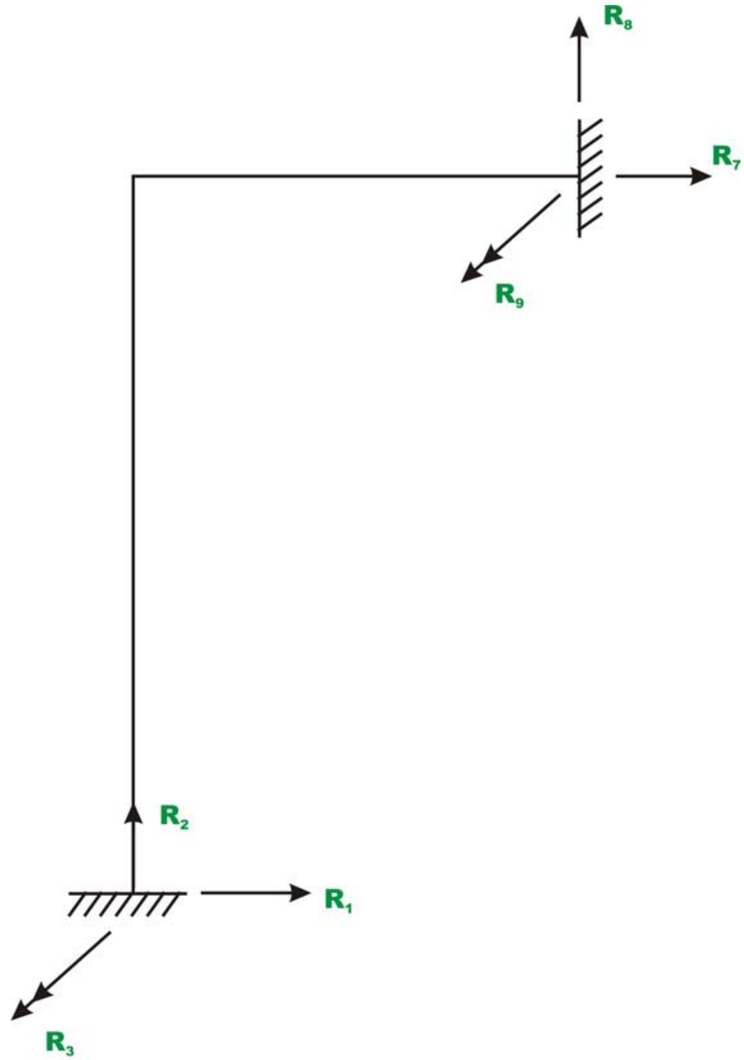

Fig. 30.4e Support Reactions.

The load vector corresponding to unconstrained degrees of freedom is (vide 30.4d),

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} \qquad (4)$$

In the given frame constraint degrees of freedom are $u_1, u_2, u_3, u_7, u_8, u_9$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load-displacement relationship for only unconstrained degree of freedom,

$$\begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} = 10^3 \begin{bmatrix} 2001.5 & 0 & 4.44 \\ 0 & 1338.3 & 10 \\ 4.44 & 10 & 44.44 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (5)$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 6.28\times10^{-6} \\ -1.695\times10^{-5} \\ -0.13\times10^{-3} \end{Bmatrix} \qquad (6)$$

$u_4 = 6.28\times10^{-6}$ m., $\quad u_5 = -1.695\times10^{-5}$

Let $R_1, R_2, R_3, R_7, R_8, R_9$ be the support reactions along degrees of freedom $1, 2, 3, 7, 8, 9$ respectively (vide Fig. 30.4e). Support reactions are calculated by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} = \begin{Bmatrix} p_1^F \\ p_2^F \\ p_3^F \\ p_7^F \\ p_8^F \\ p_9^F \end{Bmatrix} + 10^3 \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1.48 & 0 & -4.44 \\ - & -1333.3 & 0 \\ 4.44 & 0 & 8.88 \\ -2000 & 0 & 0 \\ 0 & -5 & -10 \\ 0 & 10 & 13.33 \end{bmatrix} \end{array} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} = \begin{Bmatrix} -12 \\ 0 \\ 18 \\ 0 \\ 24 \\ -24 \end{Bmatrix} + \begin{Bmatrix} 0.57 \\ 22.59 \\ -1.14 \\ -12.57 \\ 1.40 \\ -1.92 \end{Bmatrix} = \begin{Bmatrix} -11.42 \\ 22.59 \\ 16.85 \\ -12.57 \\ 25.40 \\ -25.92 \end{Bmatrix} \qquad (7)$$

Example 30.2

Analyse the rigid frame shown in Fig 30.5a by direct stiffness matrix method. Assume $E = 200$ GPa ; $I_{ZZ} = 1.33 \times 10^{-5}$ m^4 and $A = 0.01 m^2$. The flexural rigidity EI and axial rigidity EA are the same for all beams.

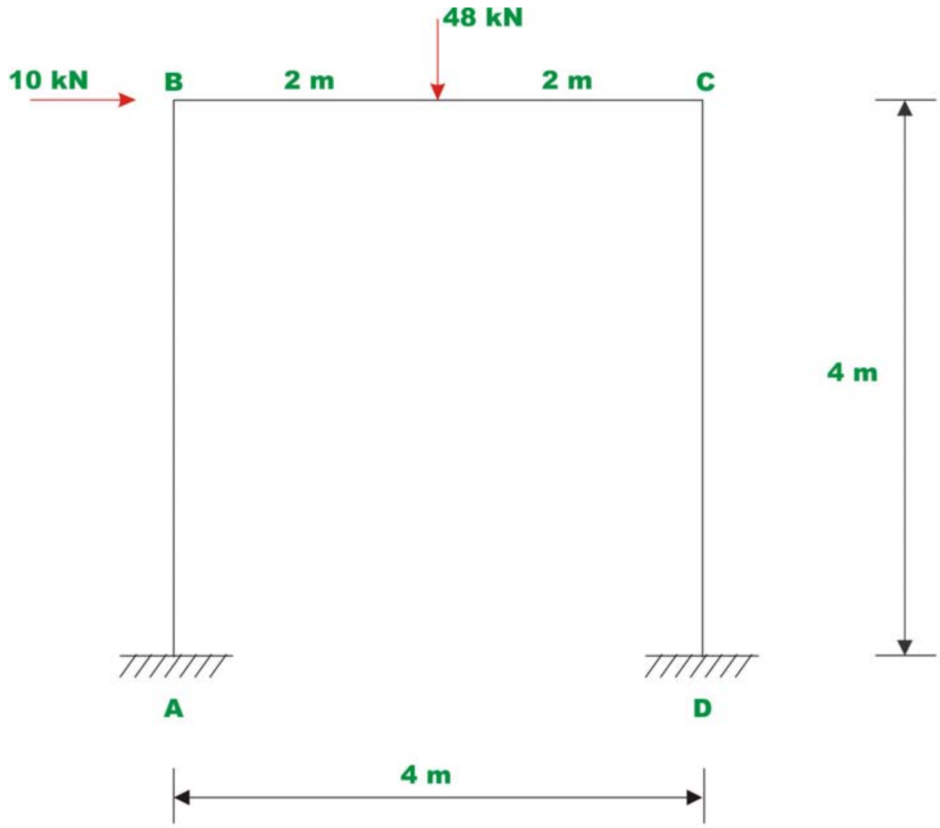

Fig. 30.5a Rigid Frame of Example 30.2

Solution:

The plane frame is divided in to three beam elements as shown in Fig. 30.5b. The numbering of joints and members are also shown in Fig. 30.5b. The possible degrees of freedom at nodes are also shown in the figure. The origin of the global co- ordinate system is taken at A (node 1).

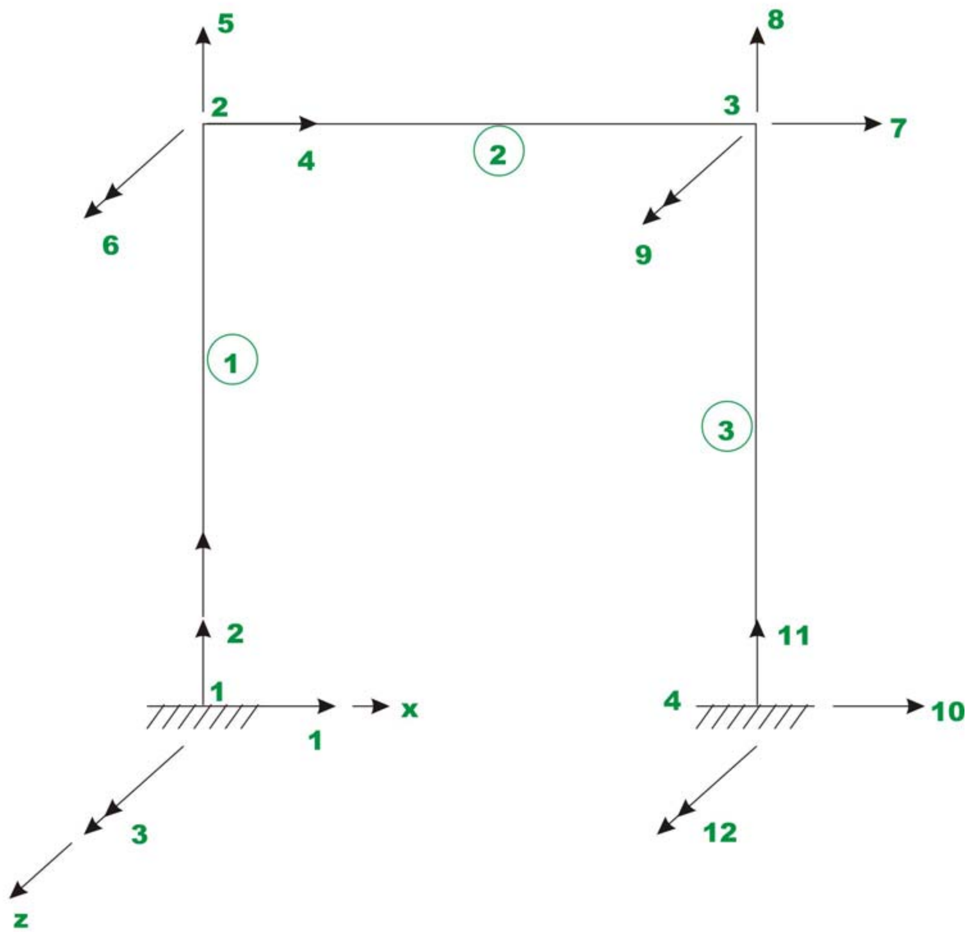

Fig. 30.5b Node and Member numbering.

Now formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. In the present case three degrees of freedom are considered at each node.

Member 1: $L = 4\ m$; $\theta = 90°$; node points 1-2 ; $l = \frac{x_2 - x_1}{L} = 0$ and $m = \frac{y_2 - y_1}{L} = 1$.

The following terms are common for all elements.

$$\frac{AE}{L} = 5 \times 10^5 \text{ kN/m}; \quad \frac{6EI}{L^2} = 9.998 \times 10^2 \text{kN}$$

$$\frac{12EI}{L^3} = 4.999 \times 10^2 \text{ kN/m}; \quad \frac{4EI}{L} = 2.666 \times 10^3 \text{kN.m}$$

$$\left[k^1\right]=[T]^T[k'][T]$$

$$
= \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{cccccc}
0.50\times10^3 & 0 & -1\times10^3 & -0.50\times10^3 & 0 & -1\times10^3 \\
0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\
-1\times10^3 & 0 & 2.66\times10^3 & 1\times10^3 & 0 & 1.33\times10^3 \\
-0.50\times10^3 & 0 & 1\times10^3 & 0.50\times10^3 & 0 & 1\times10^3 \\
0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\
-1\times10^3 & 0 & 1.33\times10^3 & 1\times10^3 & 0 & 2.66\times10^3
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

(1)

Member 2: $L=4\ m$; $\theta=0°$ node points 2-3; $l=1$ and $m=0$.

$$\left[k^2\right]=[T]^T[k'][T]$$

$$
= \begin{array}{c}
\begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{cccccc}
5.0\times10^5 & 0 & 0 & -5.0\times10^6 & 0 & 0 \\
0 & 0.5\times10^3 & 1\times10^3 & 0 & -0.5\times10^3 & 1\times10^3 \\
0 & 1\times10^3 & 2.666\times10^3 & 0 & -1\times10^3 & 1.33\times10^3 \\
-5.0\times10^6 & 0 & 0 & 5.0\times10^6 & 0 & 0 \\
0 & -0.5\times10^3 & -1\times10^3 & 0 & 0.5\times10^3 & -1\times10^3 \\
0 & 1\times10^3 & 1.33\times10^3 & 0 & -1\times10^3 & 2.666\times10^3
\end{array}\right]
\end{array}
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

(2)

Member 3: $L=4\ m$; $\theta=270°$; node points 3-4 ; $l=\dfrac{x_2-x_1}{L}=0$ and $m=\dfrac{y_2-y_1}{L}=-1$.

$$[k^3] = [T]^T [k'][T]$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
= & \left[\begin{array}{cccccc}
0.50\times10^3 & 0 & 1\times10^3 & -0.50\times10^3 & 0 & 1\times10^3 \\
0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\
1\times10^3 & 0 & 2.66\times10^3 & -1\times10^3 & 0 & 1.33\times10^3 \\
-0.50\times10^3 & 0 & -1\times10^3 & 0.50\times10^3 & 0 & -1\times10^3 \\
0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\
1\times10^3 & 0 & 1.33\times10^3 & -1\times10^3 & 0 & 2.66\times10^3
\end{array}\right]
\begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
\tag{3}
$$

The assembled global stiffness matrix $[K]$ is of the order 12×12. Carrying out assembly in the usual manner, we get,

$$
[K] = 10^3 \times \left[\begin{array}{ccc|cccccc|ccc}
0.50 & 0 & -1.0 & -0.50 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 500 & 0 & 0 & -500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.0 & 0 & 2.66 & 1.0 & 0 & 1.33 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-0.50 & 0 & 1.0 & 500.5 & 0 & 1.0 & -500 & 0 & 0 & 0 & 0 & 0 \\
0 & -500 & 0 & 0 & 500.5 & 1.0 & 0 & -0.50 & 1.0 & 0 & 0 & 0 \\
-1.0 & 0 & 1.33 & 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 & 0 & 0 & 0 \\
0 & 0 & 0 & -500 & 0 & 0 & 500.5 & 0 & 1.0 & -0.5 & 0 & 1.0 \\
0 & 0 & 0 & 0 & -0.5 & -1.0 & 0 & 500.5 & -1.0 & 0 & -500 & 0 \\
0 & 0 & 0 & 0 & 1.0 & 1.33 & 1.0 & -1.0 & 5.33 & -1.0 & 0 & 1.33 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & -1.0 & 0.5 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -500 & 0 & 0 & 500 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 1.33 & -1 & 0 & 2.66
\end{array}\right]
\tag{4}
$$

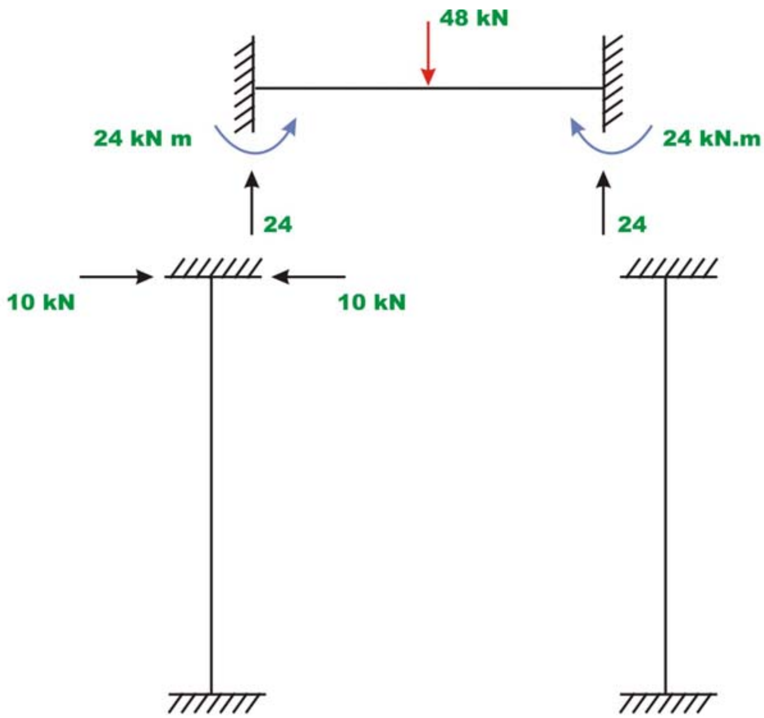

Fig . 30.5c Fixed end action due to external load.

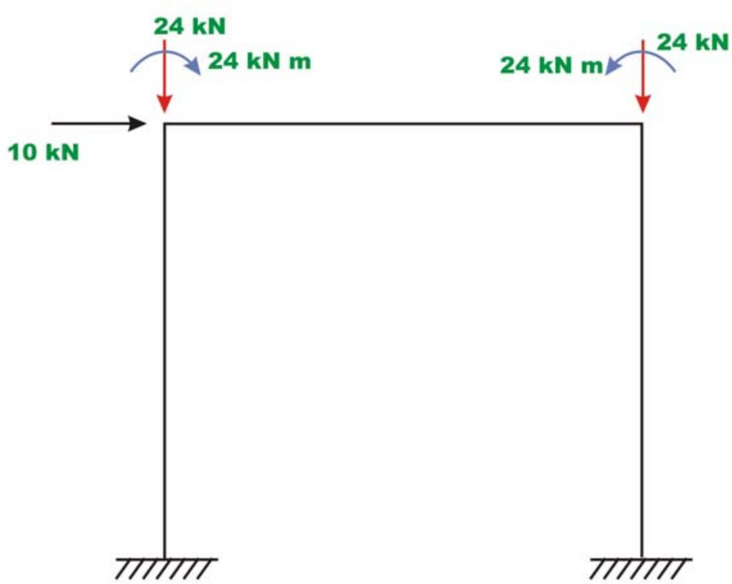

Fig. 30.5d Equivalent joint loads.

The load vector corresponding to unconstrained degrees of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} \tag{5}$$

In the given frame, constraint (known) degrees of freedom are $u_1, u_2, u_3, u_{10}, u_{11}, u_{12}$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load displacement relationship,

$$\begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} = 10^3 \begin{bmatrix} 500.5 & 0 & 1.0 & -500 & 0 & 0 \\ 0 & 500.5 & 1.0 & 0 & -0.5 & 1.0 \\ 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 \\ -500 & 0 & 0 & 500.5 & 0 & 1 \\ 0 & -0.5 & -1 & 0 & 500.5 & -1 \\ 0 & 1 & 1.33 & 1 & -1 & 5.33 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} \tag{6}$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 1.43 \times 10^{-2} \\ -3.84 \times 10^{-5} \\ -8.14 \times 10^{-3} \\ 1.43 \times 10^{-2} \\ -5.65 \times 10^{-5} \\ 3.85 \times 10^{-3} \end{Bmatrix} \tag{7}$$

Let $R_1, R_2, R_3, R_{10}, R_{11}, R_{12}$ be the support reactions along degrees of freedom $1, 2, 3, 10, 11, 12$ respectively. Support reactions are calculated by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} p_1^{\ F} \\ p_2^{\ F} \\ p_3^{\ F} \\ p_{10}^{\ F} \\ p_{11}^{\ F} \\ p_{12}^{\ F} \end{Bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} -0.5 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 & 0 & 0 \\ 1.0 & 0 & 1.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & -1.0 \\ 0 & 0 & 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 1.33 \end{bmatrix} \end{array} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} = \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} \tag{8}$$

Summary

In this lesson, the analysis of plane frame by the direct stiffness matrix method is discussed. Initially, the stiffness matrix of the plane frame member is derived in its local co-ordinate axes and then it is transformed to global co-ordinate system. In the case of plane frames, members are oriented in different directions and hence before forming the global stiffness matrix it is necessary to refer all the member stiffness matrices to the same set of axes. This is achieved by transformation of forces and displacements to global co-ordinate system. In the end, a few problems are solved to illustrate the methodology.