

Module

4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson 27

The Direct Stiffness Method: Beams

Instructional Objectives

After reading this chapter the student will be able to

1. Derive member stiffness matrix of a beam element.
2. Assemble member stiffness matrices to obtain the global stiffness matrix for a beam.
3. Write down global load vector for the beam problem.
4. Write the global load-displacement relation for the beam.

27.1 Introduction.

In chapter 23, a few problems were solved using stiffness method from fundamentals. The procedure adopted therein is not suitable for computer implementation. In fact the load displacement relation for the entire structure was derived from fundamentals. This procedure runs into trouble when the structure is large and complex. However this can be much simplified provided we follow the procedure adopted for trusses. In the case of truss, the stiffness matrix of the entire truss was obtained by assembling the member stiffness matrices of individual members.

In a similar way, one could obtain the global stiffness matrix of a continuous beam from assembling member stiffness matrix of individual beam elements. Towards this end, we break the given beam into a number of beam elements. The stiffness matrix of each individual beam element can be written very easily. For example, consider a continuous beam $ABCD$ as shown in Fig. 27.1a. The given continuous beam is divided into three beam elements as shown in Fig. 27.1b. It is noticed that, in this case, nodes are located at the supports. Thus each span is treated as an individual beam. However sometimes it is required to consider a node between support points. This is done whenever the cross sectional area changes suddenly or if it is required to calculate vertical or rotational displacements at an intermediate point. Such a division is shown in Fig. 27.1c. If the axial deformations are neglected then each node of the beam will have two degrees of freedom: a vertical displacement (corresponding to shear) and a rotation (corresponding to bending moment). In Fig. 27.1b, numbers enclosed in a circle represents beam numbers. The beam $ABCD$ is divided into three beam members. Hence, there are four nodes and eight degrees of freedom. The possible displacement degrees of freedom of the beam are also shown in the figure. Let us use lower numbers to denote unknown degrees of freedom (unconstrained degrees of freedom) and higher numbers to denote known (constrained) degrees of freedom. Such a method of identification is adopted in this course for the ease of imposing boundary conditions directly on the structure stiffness matrix. However, one could number sequentially as shown in Fig. 27.1d. This is preferred while solving the problem on a computer.

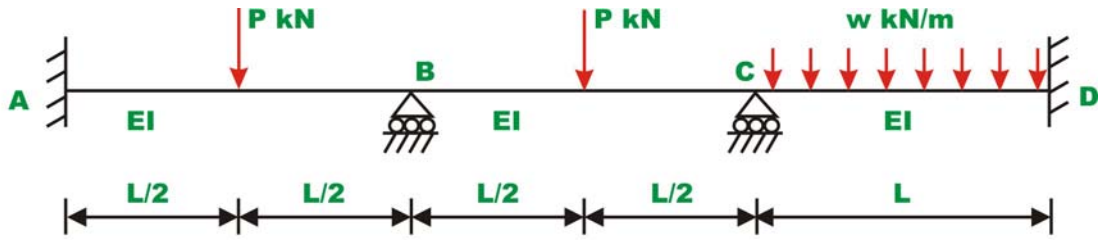


Fig 27.1a Continuous beam

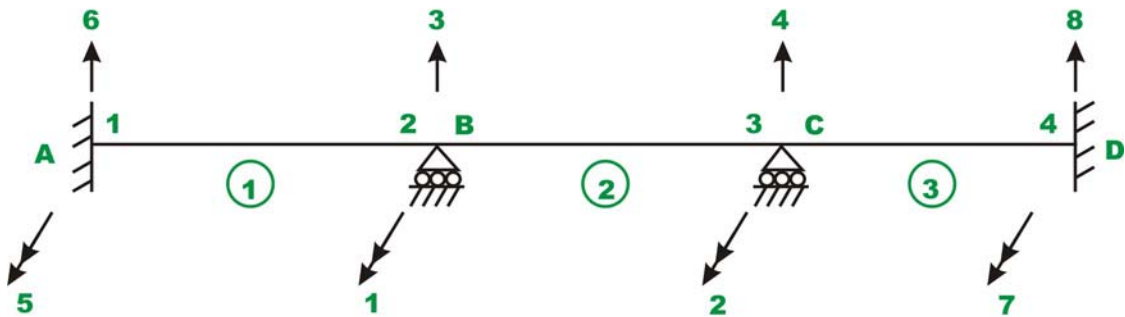


Fig. 27.1b Member and node numbering

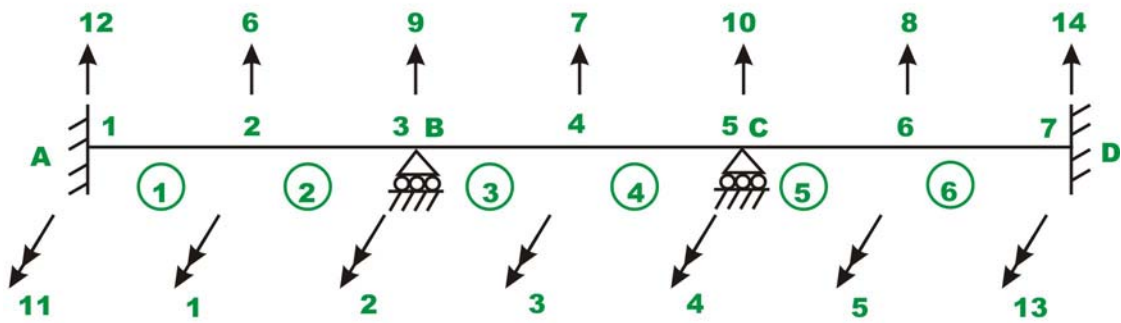


Fig. 27.1c Member and node numbering

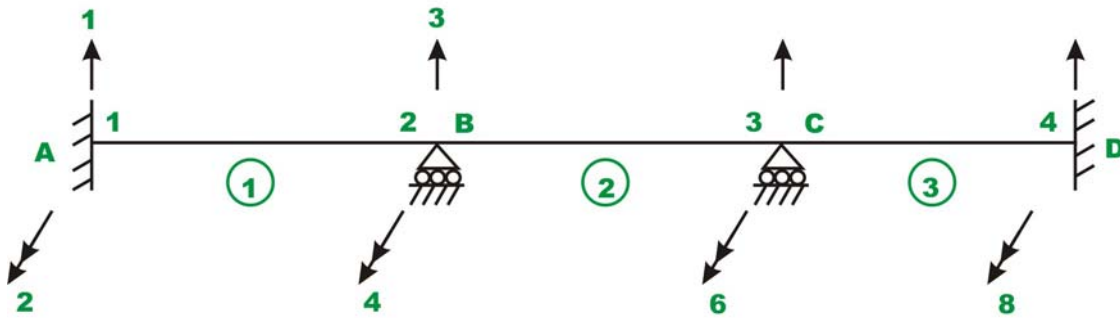


Fig 27.1d Member and node numbering

In the above figures, single headed arrows are used to indicate translational and double headed arrows are used to indicate rotational degrees of freedom.

27.2 Beam Stiffness Matrix.

Fig. 27.2 shows a prismatic beam of a constant cross section that is fully restrained at ends in local orthogonal co-ordinate system $x'y'z'$. The beam ends are denoted by nodes j and k . The x' axis coincides with the centroidal axis of the member with the positive sense being defined from j to k . Let L be the length of the member, A area of cross section of the member and I_{zz} is the moment of inertia about z' axis.

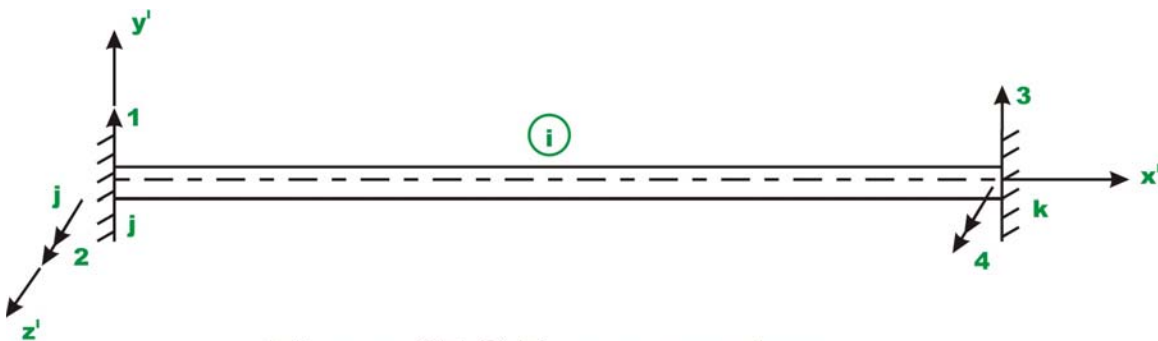
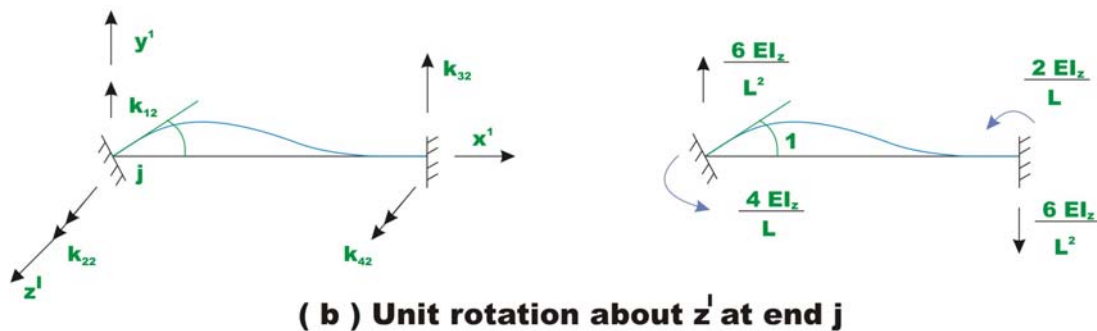
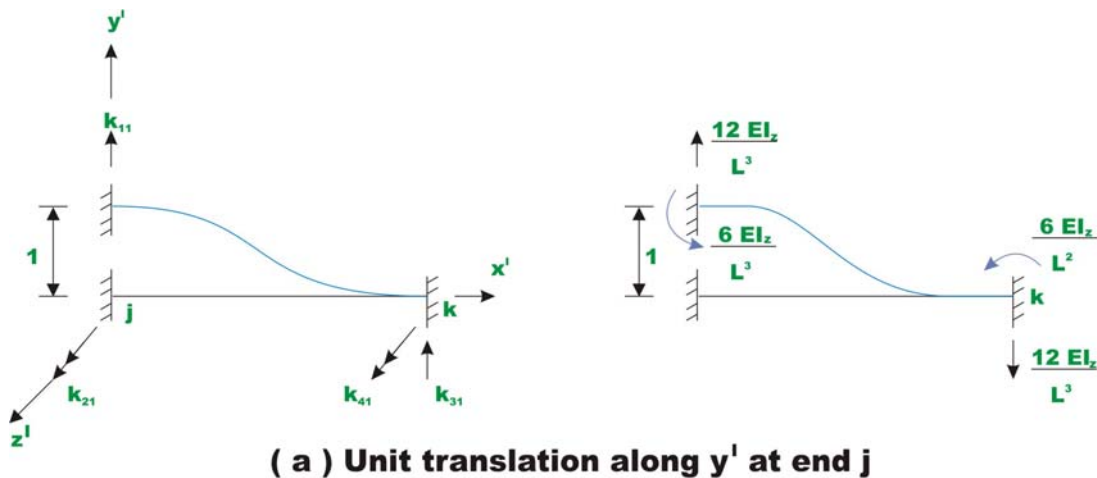


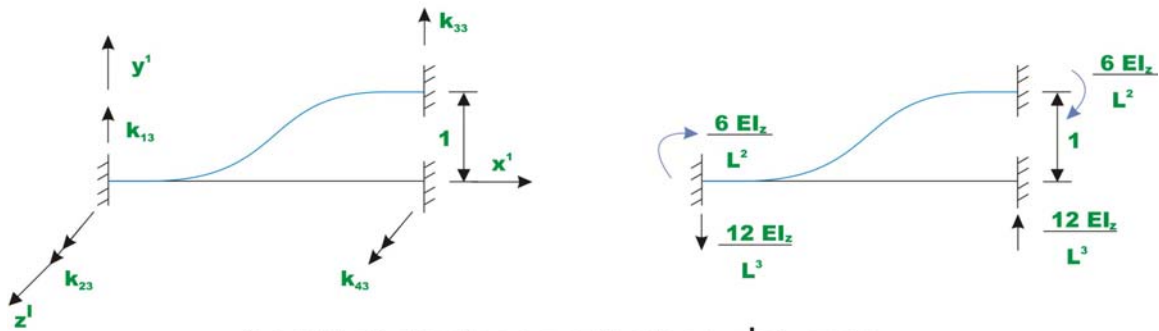
Figure 27.2 Beam member

Two degrees of freedom (one translation and one rotation) are considered at each end of the member. Hence, there are four possible degrees of freedom for this member and hence the resulting stiffness matrix is of the order 4×4 . In this method counterclockwise moments and counterclockwise rotations are taken as positive. The positive sense of the translation and rotation are also shown in the figure. Displacements are considered as positive in the direction of the coordinate axis. The elements of the stiffness matrix indicate the forces exerted on

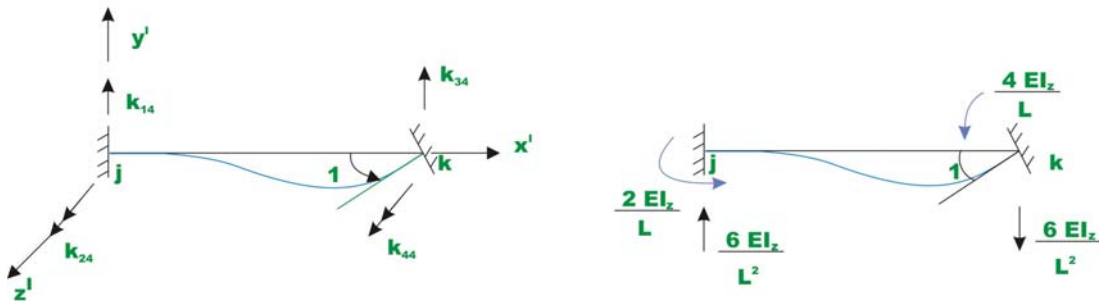
the member by the restraints at the ends of the member when unit displacements are imposed at each end of the member. Let us calculate the forces developed in the above beam member when unit displacement is imposed along each degree of freedom holding all other displacements to zero. Now impose a unit displacement along y' axis at j end of the member while holding all other displacements to zero as shown in Fig. 27.3a. This displacement causes both shear and moment in the beam. The restraint actions are also shown in the figure. By definition they are elements of the member stiffness matrix. In particular they form the first column of element stiffness matrix.

In Fig. 27.3b, the unit rotation in the positive sense is imposed at j end of the beam while holding all other displacements to zero. The restraint actions are shown in the figure. The restraint actions at ends are calculated referring to tables given in lesson ...





(c) Unit displacement along y' at end k



(d) Unit rotation about z' at end k

Fig. 27.3 Computation of beam stiffness matrix

In Fig. 27.3c, unit displacement along y' axis at end k is imposed and corresponding restraint actions are calculated. Similarly in Fig. 27.3d, unit rotation about z' axis at end k is imposed and corresponding stiffness coefficients are calculated. Hence the member stiffness matrix for the beam member is

$$[k] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \quad (27.1)$$

The stiffness matrix is symmetrical. The stiffness matrix is partitioned to separate the actions associated with two ends of the member. For continuous beam problem, if the supports are unyielding, then only rotational degree of freedom

shown in Fig. 27.4 is possible. In such a case the first and the third rows and columns will be deleted. The reduced stiffness matrix will be,

$$[k] = \begin{bmatrix} \frac{4EI_z}{L} & \frac{2EI_z}{L} \\ \frac{2EI_z}{L} & \frac{4EI_z}{L} \end{bmatrix} \quad (27.2)$$

Instead of imposing unit displacement along y' at j end of the member in Fig. 27.3a, apply a displacement u'_1 along y' at j end of the member as shown in Fig. 27.5a, holding all other displacements to zero. Let the restraining forces developed be denoted by q_{11}, q_{21}, q_{31} and q_{41} .

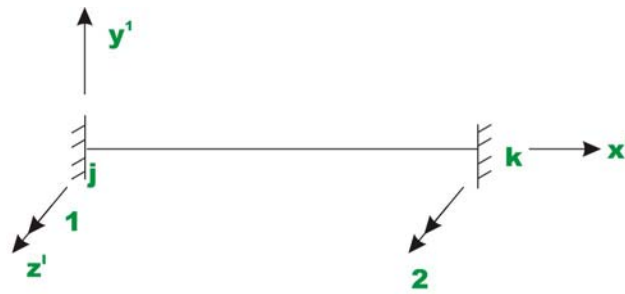


Fig. 27.4

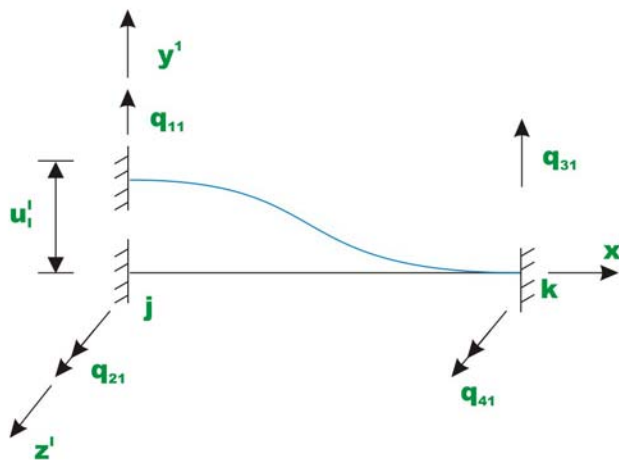


Fig. 27.5 (a)

The forces are equal to,

$$q_{11} = k_{11}u'_1 \quad ; \quad q_{21} = k_{21}u'_1 \quad ; \quad q_{31} = k_{31}u'_1 \quad ; \quad q_{41} = k_{41}u'_1 \quad (27.3)$$

Now, give displacements u'_1, u'_2, u'_3 and u'_4 simultaneously along displacement degrees of freedom 1,2,3 and 4 respectively. Let the restraining forces developed at member ends be q_1, q_2, q_3 and q_4 respectively as shown in Fig. 27.5b along respective degrees of freedom. Then by the principle of superposition, the force displacement relationship can be written as,

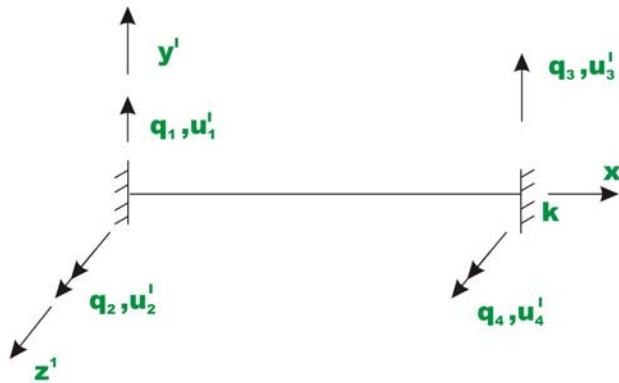


Fig. 27.5 (b) Force - displacement relation

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{bmatrix} \quad (27.4)$$

This may also be written in compact form as,

$$\{q\} = [k] \{u\} \quad (27.5)$$

27.3 Beam (global) Stiffness Matrix.

The formation of structure (beam) stiffness matrix from its member stiffness matrices is explained with help of two span continuous beam shown in Fig. 27.6a. Note that no loading is shown on the beam. The orthogonal co-ordinate system xyz denotes the global co-ordinate system.

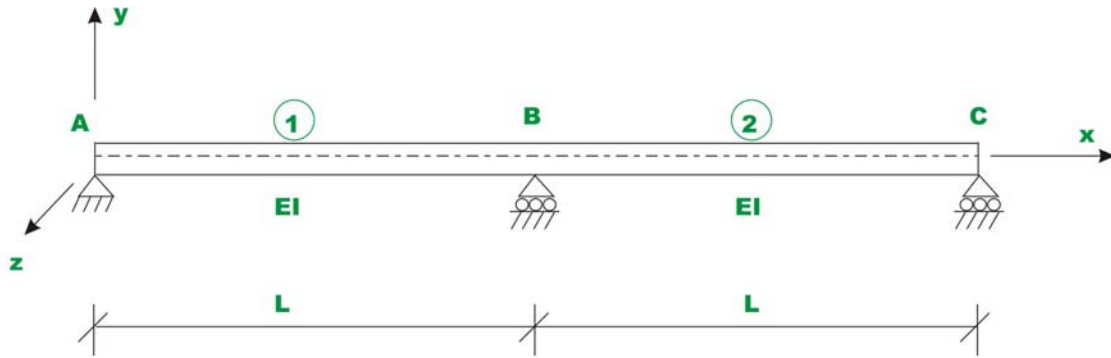


Fig. 27.6a Continuous beam

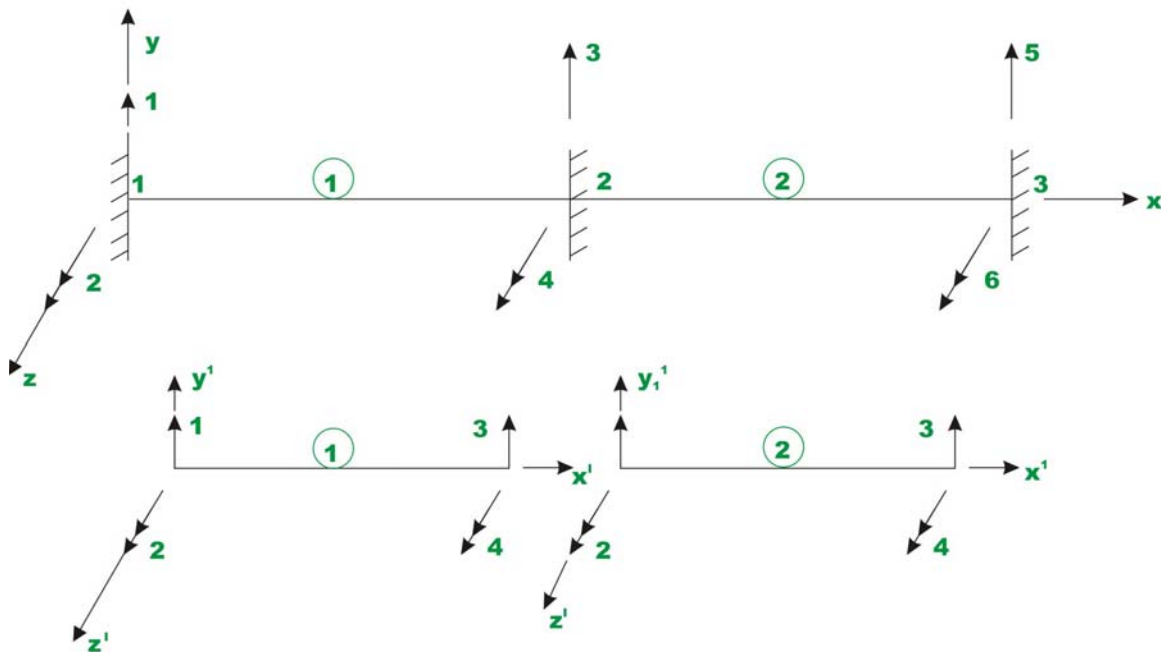


Fig. 27.6 b

For the case of continuous beam, the x - and x' - axes are collinear and other axes (y and y' , z and z') are parallel to each other. Hence it is not required to transform member stiffness matrix from local co-ordinate system to global co

ordinate system as done in the case of trusses. For obtaining the global stiffness matrix, first assume that all joints are restrained. The node and member numbering for the possible degrees of freedom are shown in Fig 27.6b. The continuous beam is divided into two beam members. For this member there are six possible degrees of freedom. Also in the figure, each beam member with its displacement degrees of freedom (in local co ordinate system) is also shown. Since the continuous beam has the same moment of inertia and span, the member stiffness matrix of element 1 and 2 are the same. They are,

$$\begin{array}{r}
 \text{Global d.o.f} \quad 1 \quad 2 \quad 3 \quad 4 \\
 \text{Local d.o.f} \quad 1 \quad 2 \quad 3 \quad 4
 \end{array}
 \quad [k'] = \begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} \end{bmatrix} \begin{array}{l} 1 \quad 1 \\ 2 \quad 2 \\ 3 \quad 3 \\ 4 \quad 4 \end{array} \quad (27.6a)$$

$$\begin{array}{r}
 \text{Global d.o.f} \quad 3 \quad 4 \quad 5 \quad 6 \\
 \text{Local d.o.f} \quad 1 \quad 2 \quad 3 \quad 4
 \end{array}
 \quad [k^2] = \begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} & k^2_{14} \\ k^2_{21} & k^2_{22} & k^2_{23} & k^2_{24} \\ k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{bmatrix} \begin{array}{l} 1 \quad 3 \\ 2 \quad 4 \\ 3 \quad 5 \\ 4 \quad 6 \end{array} \quad (27.6b)$$

The local and the global degrees of freedom are also indicated on the top and side of the element stiffness matrix. This will help us to place the elements of the element stiffness matrix at the appropriate locations of the global stiffness matrix. The continuous beam has six degrees of freedom and hence the stiffness matrix is of the order 6×6 . Let $[K]$ denotes the continuous beam stiffness matrix of order 6×6 . From Fig. 27.6b, $[K]$ may be written as,

$$\begin{array}{c}
 \text{Member AB (1)} \\
 [K] = \left[\begin{array}{cc|cc|cc}
 k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 \\
 k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 \\
 \hline
 k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \\
 k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 \\
 \hline
 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\
 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2
 \end{array} \right] \quad (27.7) \\
 \text{Member BC (2)}
 \end{array}$$

The 4×4 upper left hand section receives contribution from member 1 (AB) and 4×4 lower right hand section of global stiffness matrix receives contribution from member 2. The element of the global stiffness matrix corresponding to global degrees of freedom 3 and 4 [overlapping portion of equation(27.7)] receives element from both members 1 and 2.

27.4 Formation of load vector.

Consider a continuous beam ABC as shown in Fig. 27.7.

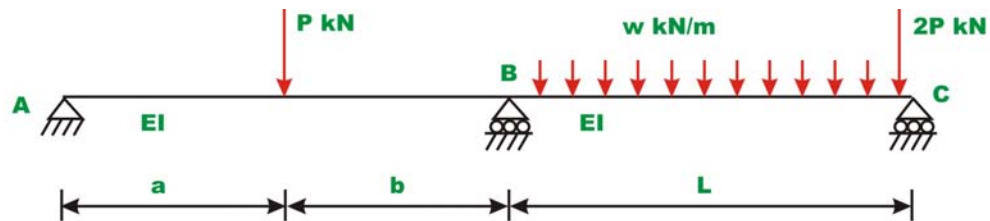
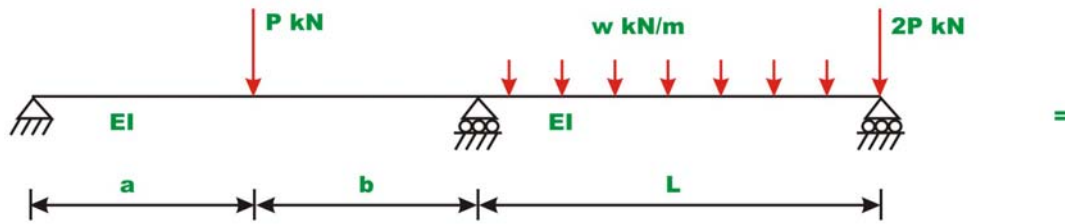
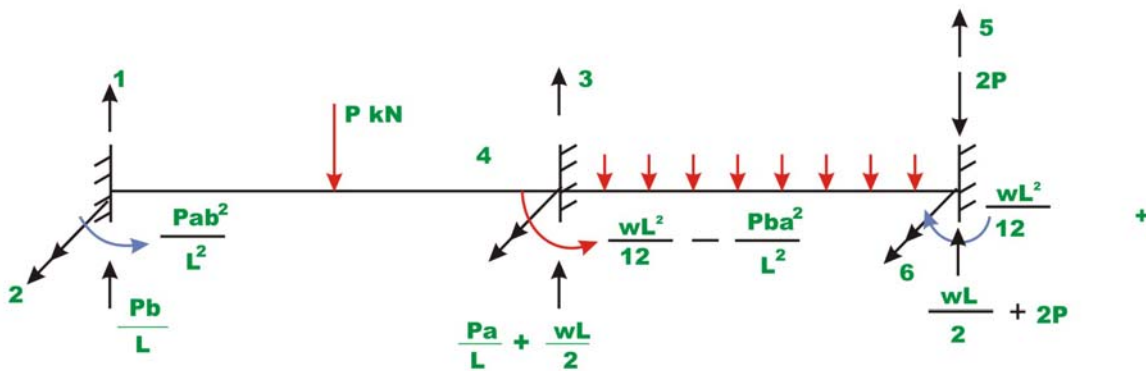


Fig.27.7

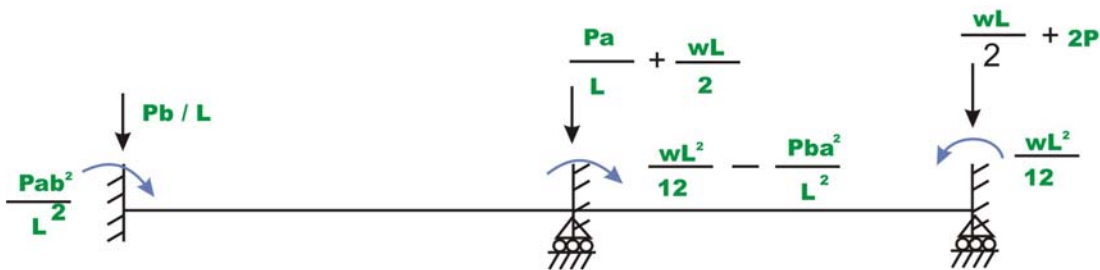
We have two types of load: member loads and joint loads. Joint loads could be handled very easily as done in case of trusses. Note that stiffness matrix of each member was developed for end loading only. Thus it is required to replace the member loads by equivalent joint loads. The equivalent joint loads must be evaluated such that the displacements produced by them in the beam should be the same as the displacements produced by the actual loading on the beam. This is evaluated by invoking the method of superposition.



(a) Actual beam with loading



(b) Reaction in the restrained beam



(c) Equivalent joint loads

Fig. 27.8

The loading on the beam shown in Fig. 27.8(a), is equal to the sum of Fig. 27.8(b) and Fig. 27.8(c). In Fig. 27.8(c), the joints are restrained against displacements and fixed end forces are calculated. In Fig. 27.8(c) these fixed end actions are shown in reverse direction on the actual beam without any load. Since the beam in Fig. 27.8(b) is restrained (fixed) against any displacement, the displacements produced by the joint loads in Fig. 27.8(c) must be equal to the displacement produced by the actual beam in Fig. 27.8(a). Thus the loads shown

in Fig. 27.8(c) are the equivalent joint loads .Let, p_1, p_2, p_3, p_4, p_5 and p_6 be the equivalent joint loads acting on the continuous beam along displacement degrees of freedom 1,2,3,4,5 and 6 respectively as shown in Fig. 27.8(b). Thus the global load vector is,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} -\frac{Pb}{L} \\ -\frac{Pab^2}{L^2} \\ -\left(\frac{Pa}{L} + \frac{wL}{2}\right) \\ -\left(\frac{wL^2}{12} - \frac{Pba^2}{L^2}\right) \\ -\left(\frac{wL}{2} + 2P\right) \\ \frac{wL^2}{12} \end{Bmatrix} \quad (27.8)$$

27.5 Solution of equilibrium equations

After establishing the global stiffness matrix and load vector of the beam, the load displacement relationship for the beam can be written as,

$$\{P\} = [K]\{u\} \quad (27.9)$$

where $\{P\}$ is the global load vector, $\{u\}$ is displacement vector and $[K]$ is the global stiffness matrix. This equation is solved exactly in the similar manner as discussed in the lesson 24. In the above equation some joint displacements are known from support conditions. The above equation may be written as

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \quad (27.10)$$

where $\{p_k\}$ and $\{u_k\}$ denote respectively vector of known forces and known displacements. And $\{p_u\}$, $\{u_u\}$ denote respectively vector of unknown forces and unknown displacements respectively. Now expanding equation (27.10),

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \quad (27.11a)$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \quad (27.11b)$$

Since $\{u_k\}$ is known, from equation 27.11(a), the unknown joint displacements can be evaluated. And support reactions are evaluated from equation (27.11b), after evaluating unknown displacement vector.

Let R_1, R_3 and R_5 be the reactions along the constrained degrees of freedom as shown in Fig. 27.9a. Since equivalent joint loads are directly applied at the supports, they also need to be considered while calculating the actual reactions. Thus,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = - \begin{Bmatrix} p_1 \\ p_3 \\ p_5 \end{Bmatrix} + [K_{21}]\{u_u\} \quad (27.12)$$

The reactions may be calculated as follows. The reactions of the beam shown in Fig. 27.9a are equal to the sum of reactions shown in Fig. 27.9b, Fig. 27.9c and Fig. 27.9d.

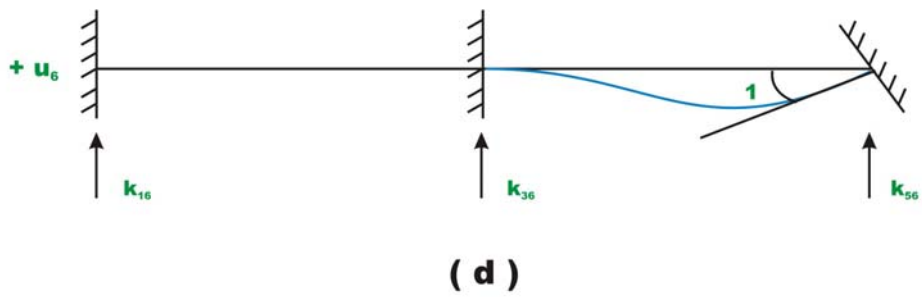
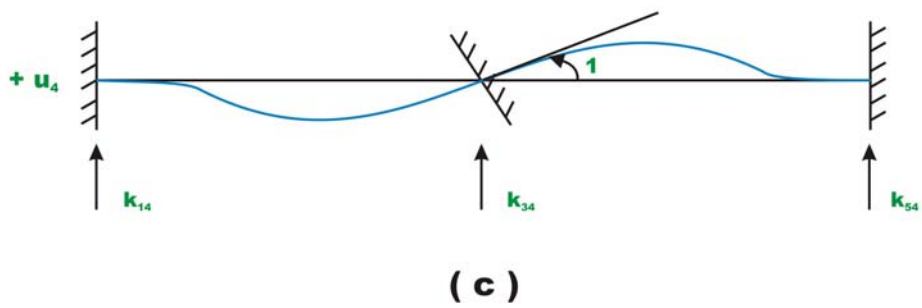
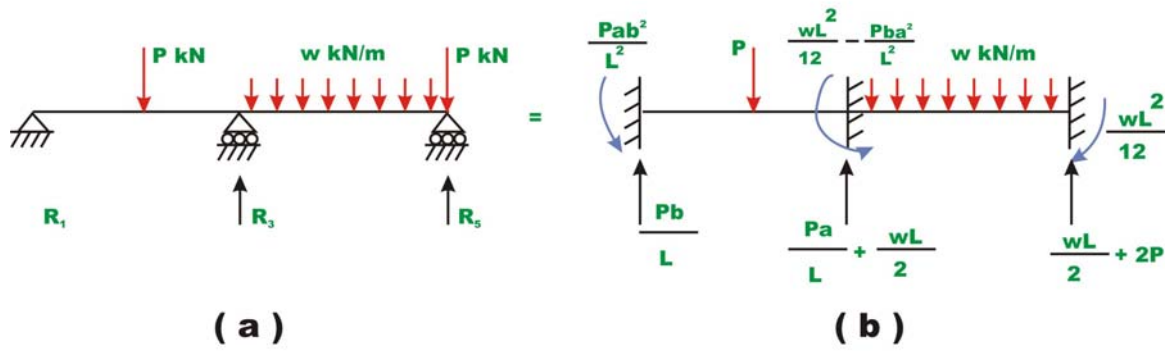


Fig. 27.9

From the method of superposition,

$$R_1 = \frac{Pb}{L} + K_{14}u_4 + K_{16}u_6 \quad (27.13a)$$

$$R_3 = \frac{Pa}{L} + K_{34}u_4 + K_{36}u_6 \quad (27.13b)$$

$$R_5 = \frac{wL}{2} + 2P + K_{54}u_4 + K_{56}u_6 \quad (27.13c)$$

or

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \quad (27.14a)$$

Equation (27.14a) may be written as,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \quad (27.14b)$$

Member end actions q_1, q_2, q_3, q_4 are calculated as follows. For example consider the first element 1.

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \frac{Pb}{L} \\ \frac{Pab^2}{L^2} \\ \frac{Pa}{L} \\ \frac{L}{Pa^2b} \\ -\frac{L^2}{L^2} \end{Bmatrix} + [K]_{element1} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} \quad (27.16)$$

In the next lesson few problems are solved to illustrate the method so far discussed.

Summary

In this lesson the beam element stiffness matrix is derived from fundamentals. Assembling member stiffness matrices, the global stiffness matrix is generated. The global load vector is defined. The global load-displacement relation is written for the complete beam structure. The procedure to impose boundary conditions on the load-displacement relation is discussed. With this background, one could analyse continuous beam by the direct stiffness method.