Lesson 9

1. What is the characteristic equation?

   The characteristic equation arises during the solution of an eigen value problem: $Ax = \lambda x$. This problem can be restated as $(A- \lambda I)x = 0$. For this system to have non trivial solutions $\det (A - \lambda I) = 0$. This scalar equation, which is a polynomial in λ is known as the characteristic equation. Its solutions are the eigen values of A.

2. Why are numerical methods necessary for solving the characteristic equation?

   Numerical methods are necessary for solving the characteristic equation because the coefficients of the powers of λ in the characteristic equation are very sensitive to minor perturbations. Thus minor perturbations in the elements of A, which should cause minor differences in the eigen values, cause large differences in the roots as obtained from the characteristic equation. In addition, if the characteristic equation is a polynomial of order 4, which it would be for a matrix of size greater than 4 x 4, no analytical solutions to the characteristic equation can be found. This is because according to Abel's theorem, analytical solutions are not possible for polynomials of order greater than 4.

3. What is Gershgorin's theorem and why is it useful?

   Gershorin's theorem provides upper and lower bounds on the eigen values of a matrix. It is useful because these bounds can be calculated even before the eigen values themselves are evaluated. Since evaluating the eigen values of large matrices is often expensive and time consuming, these bounds can be used to obtain conservative estimates of the values of the maximum and minimum eigen values with little computational expense.

4. What is meant by stability of the eigen values of a matrix? Are the eigen values of a symmetric matrix stable?

   If the perturbations to the components of matrix A, with the perturbed matrix A being defined as $A + \varepsilon B$, $\varepsilon \rightarrow 0$, result in changes in the eigen values of order ε, then the eigen values of the matrix A are said to be stable. It can be shown that the eigen values of a symmetric matrix are always stable.

5. What is the power method?

   The power method is an iterative method to find the eigen values of a matrix. It assumes that the eigen vectors of a n x n matrix are linearly independent, and starting with an arbitrary vector with dimension n, involves generation of a sequence of vectors by repeated pre-multiplication of the current iteration vector by the matrix. This process eventually leads to the sequence converging to the eigen vector corresponding to the largest eigen value. The Rayleigh's quotient can then be used to find the largest eigen value.

6. What is Rayleigh's quotient?

Rayleigh's quotient is a method to calculate the eigen value of a matrix if the corresponding eigen vector is known. For example if the k^{th} eigen vector of the matrix A, z_k is known, then the k^{th} eigen value, λ_k can be calculated as: $\lambda_k = \frac{z_k^T A z_k}{z_k^T z_k}$.

7. What is the inverse power method?

The inverse power method is an iterative method to find the smallest eigen value and corresponding eigen vector of a matrix A. It is very similar to the power method, but involves iterations that involve multiplication of the iterates by A^{-1} rather than A.