

## Lesson 8

### 1. What is an overdetermined linear system?

An overdetermined system is a linear system in which there are more equations than unknowns: the system  $Ax = b$  is such that the number of rows of  $A$ , say  $m$ , is greater than the number of columns of  $A$ ,  $n$ .

### 2. How is the solution of an overdetermined system found?

Overdetermined systems generally do not have unique solutions. Hence it is necessary to find a solution that fits the data best i.e. the solution that minimizes the residual  $r$ , where  $r = Ax - b$ .

### 3. What is the least squares method for finding the solution of an overdetermined system?

In the least squares method, the solution  $x$  is found by minimizing the  $L_2$  norm of the residual  $r = Ax - b$  i.e. that  $x$  is found that minimizes  $\|Ax - b\|_{L_2}$ . The resultant residual is orthogonal to the vector space spanned by the columns of the coefficient matrix  $A$  i.e.  $A^T r = 0$ . Thus the least squares procedure splits  $b$  into two parts:  $Ax$  and  $r$  with  $Ax$  by definition belonging to the space spanned by the columns of  $A$ .

### 4. When is the least squares solution unique?

The least squares solution is unique when the matrix  $A$  has full rank. In that case the solution can be found by solving the linear system  $A^T A x = A^T b$ . In case  $A$  does not have full rank,  $A^T A$  becomes singular in which case the above system cannot be solved. Then there are multiple solutions to the least squares problem, with the number of linearly independent solutions being equal to the dimension of the null space of  $A^T A$ .

### 5. What is the QR decomposition?

The QR decomposition is a method to decompose a matrix  $A$  with  $m$  rows and  $n$  columns, with  $m$  greater than or equal to  $n$ , into an orthogonal matrix  $Q$  and a right triangular matrix  $R$  i.e.  $A = QR$ .

### 6. When is the QR decomposition useful?

The QR decomposition is useful when the matrix  $A$  has full rank but the matrix  $A^T A$  is ill-conditioned. In such situations it is not advisable to find the least squares solution by

inverting  $A^T A$ , since the solution would not be stable. In such situations, a stable and unique solution to the least squares problem can be found using QR decomposition. If matrix  $A$  does not have full rank, i.e. it is singular, in such cases also it is possible to use modified form of the QR decomposition to find the non-unique solutions to the least squares problem.

6. What is Gram-Schmidt orthogonalization?

The Gram-Schmidt orthogonalization is a way of systematically generating a set of 'n' orthogonal vectors from a set of 'n' linearly independent vectors.