Lesson 7

1.What are iterative methods and how do they differ from direct methods?

Iterative methods are a class of methods for solving the linear system Ax = b. Unlike direct methods, based on Gauss elimination that involve a fixed number of iterations, iterative methods however do not have a fixed number of operations. They start from an initial assumption that is successively improved until a sufficiently accurate solution is obtained. Direct methods if applied to the solution of sparse systems often destroy the sparseness. Thus direct methods often cannot take advantage of the sparseness of the system in terms of reduced storage and operations. Large sparse systems are particularly suited to iterative methods since iterative methods do not introduce any additional non-zero elements to the coefficient matrix.

2. What is the main difference between the Jacobi and Gauss-Seidel method?

Both the Jacobi and Gauss-Seidel methods are iterative methods for solving the linear system Ax = b. In the Jacobi method the updated vector x is used for the computations only after all the variables (i.e. all components of the vector x) have been updated. On the other hand in the Gauss-Seidel method, the updated variables are used in the computations as soon as they are updated. Thus in the Jacobi method, during the computations for a particular iteration, the “known” values are all from the previous iteration. However in the Gauss-Seidel method, the “known” values are a mix of variable values from the previous iteration (whose values have not yet been evaluated in the current iteration), as well as variable values that have already been updated in the current iteration. Even though the Gauss-Seidel’s method uses the improved values as soon as they are computed, this does not ensure that the Gauss-Seidel’s method would converge faster than Jacobi iterations

3. What is a stationary iterative method?

A stationary iterative method is an iterative method for which the update algorithm can be written in the form:

$$\mathbf{x}^{(k+1)} = \mathbf{B}\,\mathbf{x}^{(k)} + \mathbf{c}, \quad k = 0,1,2\ldots$$

where the matrix B does not change from iteration to iteration. Both the Jacobi and Gauss-Seidel methods are stationary iterative methods.

4. Do the Jacobi and Gauss-Seidel methods always converge?

No, in common with all stationary iterative methods, the Jacobi and Gauss-Seidel methods converge only when the convergence requirement is fulfilled. The necessary and sufficient convergence requirement for a stationary iterative method defined as $\mathbf{x}^{(k+1)} = \mathbf{B}\mathbf{x}^{(k)} + \mathbf{c}$ to converge is that $\rho(\mathbf{B}) = \max_{1\le i\le n} |\lambda_i(\mathbf{B})| < 1$ where $\rho(\mathbf{B})$ is the spectral radius of B. In other words the spectral radius of B must be less than 1.0 for the method to converge. It can be shown that this condition is fulfilled for both Jacobi and Gauss Seidel if the coefficient matrix A is diagonally dominant.

5. What is meant by the rate of convergence of an iterative method?

The rate of convergence of an iterative method determines on how fast the error $\left|\mathbf{x}^{(k)} - \mathbf{x}\right|$ goes to zero as k, the number of iterations, increases. The asymptotic rate of convergence for Jacobi and Gauss-Seidel, i.e. the rate of convergence for large k, depends on the factor $R = -\log(\rho_B)$, which again depends on the spectral radius of B, ρ_B

6. What is the SOR method?

The SOR method, or the successive over-relaxation method, is an improvement to the Gauss-Seidel algorithm. It substantially increases the rate of convergence of the Gauss-Seidel method.