Lesson 4

1.What is Sylvester's criterion?

Sylvester's criterion allows determination of whether a symmetric matrix is positive definite. According to this criterion a symmetric n x n matrix A is positive definite if and only if $\det(A_k) > 0$, k =1, 2, ….n, where A_k is the k x k matrix formed by the intersection of the first k rows and k columns.

2. What is LU decomposition?

LU decomposition allows a coefficient matrix to be written as the product of a lower triangular matrix and an upper triangular matrix. If A be a n x n matrix, and A_k denotes the k x k matrix formed by the intersection of the first k rows and k columns of A, then if det (A_k) not equal to zero for k = 1,2,3….n-1, then there exists a unique lower triangular matrix L = (m_{ij}) with m_{ii} =1, i=1,2….n-1 and a unique upper triangular matrix U = (u_{ij}) so that LU = A.

3. What is the advantage of performing a LU decomposition of the coefficient matrix of a linear system?

If it is possible to decompose the coefficient matrix A into the product of a lower triangular and an upper triangular matrix, i.e. A = LU, then the system Ax = b can be written as Ly = b and Ux = y. This involves solving two triangular systems using back substitution and forward substitution. Since solving a triangular system requires $n^2/2$ operations, if the decomposition A=LU is known a priori, solving the system Ax=b would only require n^2 operations. In comparison Gauss elimination requires $n^3/3$ operations. Thus the computational cost of solving a LU decomposed system as compared to a system where full Gauss elimination is required is significantly less.

4. When can Gauss elimination be performed without any pivoting?

This can be done for a positive definite matrix. For a positive definite matrix of size n x n, each of the sub-matrices A_k that denote the k x k matrix formed by the intersection of the first k rows and k columns of A, for k = 1,2,3….n-1, are assured to have positive determinant. In other words $\det(A_k) > 0$ for k = 1,2,3….n-1. This ensures that all the pivot elements during Gaussian elimination are non-zero and hence no pivoting is required.

5. What is Crout's method?

Crout's method is a more compact method for Gauss elimination that allows determination of the L and U matrices directly and has smaller storage requirements than Gauss elimination.

6. What is Choleski Decomposition?

Choleski Decomposition is a particularly useful compact form for positive definite matrices. It is based on the results that for a symmetric positive definite matrix A there is a unique upper triangular matrix with positive diagonal elements such that A $=R^TR$. It again allows solution of the linear system Ax = b by solving two triangular systems $R^Ty = b$ and Rx = y. However the method requires computation of n square roots which makes the method somewhat expensive. If the matrix R can be found, the linear system can again be solved by two sets of substitutions, thereby reducing the cost of solution from $\frac{n^3}{3}$ for Gauss elimination to n^2. This may require computation of n square roots which makes the method somewhat expensive.