Lesson 40

1. What is the basis for approximate methods for solving integral equations?

   In most approximate methods for solving integral equations, the solution is approximated by a linear combination of suitable basis functions:

$$\tilde{y}(x) = \sum_{k=1}^{N} c_k \varphi_k(x).$$

For instance, the solution of the Fredholm equation of the second kind, $y(x) = F(x) + \lambda$ this results in $\sum_{k=1}^{N} c_k \varphi_k(x) \approx F(x) + \lambda \int_a^b K(x,\xi) \sum_{k=1}^{N} c_k \varphi_k(\xi)\, d\xi.$

Denoting $\int_a^b K(x,\xi)\varphi_k(\xi)\, d\xi = \psi_k(\xi)$, in order to adequately satisfy the integral equation, the coefficients c_k must satisfy the condition:

$$\sum_{k=1}^{N} c_k [\varphi_k(x) - \lambda \psi_k(x)] \approx F(x) \quad \forall x \in [a,b]$$

Since there are N c_k's, N conditions are required that would give rise to the necessary N equations to determine the c_k's. The various approximate methods for solution of integrals equations each correspond to different conditions for the determination of the c_k's.

2. What is the major problem that can arise with using approximate methods for solving integral equations?

There may be situations where for a particular $\tilde{y}(x) = \sum_{k=1}^{N} c_k \varphi_k(x)$, the relation

$$\sum_{k=1}^{N} c_k [\phi_k(x) - \lambda \psi_k(x)] \approx F(x) \quad \forall x \in [a,b]$$

may be satisfied to an acceptable level of precision, but $\tilde{y}(x)$ may not represent the analytical solution of the integral equation, $y(x)$, to an acceptable level of accuracy, i.e.

$$\left\| \sum_{k=1}^{N} c_k [\varphi_k(x) - \lambda \psi_k(x)] - F(x) \right\| < \varepsilon$$

where ε is an acceptably small number, but $\| y(x) - \tilde{y}(x) \| >> \varepsilon$

3. How can a more dependable estimate to the true solution be obtained using an approximate method?

A more dependable estimate of $y(x)$ can generally be obtained if in addition to $\tilde{y}(x)=\sum_{k=1}^{N} c_k \varphi_k(x)$ satisfying the condition (*), $\tilde{\tilde{y}}(x)=\sum_{k=1}^{N+1} c_k \varphi_k(x)$ also satisfies (*).

4. How can the collocation method be used to determine an approximate solution for an integral equation?

The collocation method supplies a set of N conditions for determining the N unknowns c_k by requiring that (*) be satisfied at N distinct points in the interval $[a,b]$. Restating the constraint equation as $\sum_{k=1}^{N} c_k(\phi_k(x)-\lambda\psi_k(x))-F(x)\approx 0$ the collocation method requires the constraint equation to be satisfied at N points. If the N points in $[a,b]$ where the constraint equations will be satisfied are given by $\{x_1, x_2, \ldots x_N\}$, then the N equations for determining the c_k's are:

$$\sum_{k=1}^{N} c_k z_k(x_i)=F(x_i) \quad i=1,2\ldots N$$

5. How is the Galerkin method used to determine an approximate solution for an integral equation?

An alternative method for determining the N conditions for determining the constants is to use the Galerkin method. Since in the Galerkin method, we require the residual to be orthogonal to the space of weight functions, if we denote the weight functions as the set of N linearly independent functions $\overline{\varphi}_i(x),\ i=1,2\ldots N$ the N conditions are of the form:

$$\int_a^b \sum_{k=1}^{N}(c_k z_k - F)\overline{\varphi}_i(x)dx = 0 \Rightarrow \sum_{k=1}^{N} c_k \int_a^b z_k(x)\overline{\varphi}_i(x)\,dx = \int_a^b F(x)\overline{\varphi}_i(x)\,dx$$

Thus we finally get a set of equations:

$$\mathbf{Mc}=\mathbf{b} \text{ where } m_{ij}=\int_a^b \overline{\varphi}_i(x)\,z_j(x)dx; \quad b_i=\int_a^b F(x)\overline{\varphi}_i(x)dx$$