Lesson 3

1.Why are linear systems important?

While most physcial problems are nonlinear, even nonlinear problems often require as part of their solution efficient ways of solving a linear system. Thus linear systems arise often and frequently in numerical analysis. Efficient solution of a linear system is crucial to the success of many numerical algorithms.

2. When are a set of vectors linearly independent?

The vectors x_1, x_2, x_n are said to be linearly independent if the vector y, obtained as a linear combination of x_1, x_2,…. x_n i.e. $y = c_1x_1+c_2x_2+\ldots\ldots c_nx_n = 0$ with $c_1, c_2,\ldots. c_n$ being arbitrary constants not all of which are equal to zero, is equal to zero.

3. What is a similarity transformation?

A similarity transformation involves the non-singular matrix C operating on the matrix A to yield the matrix B given by $B = C^{-1}AC$. A similarity transformation preserves the eigen values. Thus the eigen values of B are the same as those of A. However a similarity transformation transforms the eigen vectors. If x is an eigen vector of A, corresponding to the eigen value λ, the corresponding eigen vector of B (i..e the eigen vector corresponding to the same eigen value λ) is given by $x^* = C^{-1}x$.

4. What is a singular value decomposition of a matrix?

If A is a *m x n* matrix of rank *r,* then there exist a *m x m* orthogonal matrix U (i.e. $U^TU = I$), a *n x n* orthogonal matrix V (i.e. $V^TV = I$) and a *r x r* diagonal matrix D with strictly positive elements, called the singular values of A such that:

$$\mathbf{A} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^{\mathrm{T}}, \quad \mathbf{\Sigma} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$$

The decomposition of A in terms of U, V and D as described above is known as the singular value decomposition of A.

5. What are the two principal methods for solving a system of linear equations?

The first class of methods includes direct methods which involve a sequence of elimination operations that after a certain number of steps give the exact solution, in the absence of any rounding errors. The second class involves iterative methods which give a sequence of approximate solutions that converge to the true solution as the number of iterations tends to infinite.

For full matrices, direct methods are most efficient while for sparse matrices (i.e. where a large proportion of the elements are zero) iterative methods may give useful results with a fewer number of operations than direct methods. For very large sparse systems iterative methods are preferable. Thus the choice between the two methods depends on the proportion and distribution as well as the sign and size of the non-zero elements.

6. What is the band width of a matrix and what is band width reduction?

A matrix A for which $a_{ij} = 0$ if $j > i+p$ or $i > j+q$ is called a banded matrix with band width $w = p+q+1$. In such a matrix the number of non zero elements in any row or column is less than the band width w.
Band width reduction refers to the process of relabelling the variables in a linear system so as to reduce the bandwidth of the coefficient matrix. It leads to an increase in sparsity of the coefficient matrix and makes it “more diagonal” which results in a decrease in the cost of matrix inversion.

7. What is the basic principle of Gauss elimination?

The basic idea is to eliminate the unknowns in a systematic way that results in the transformation of a full coefficient matrix of a linear system into a triangular coefficient matrix.