Lesson 36

1. How can linearly independent basis functions used for the solution of linear partial differential equations?

The general linear partial differential equation can be stated as :

$$Lu = f \text{ in } D,\ \bar{L}u = g \text{ on } \partial D$$

Given an appropriate set of linearly independent basis functions $\{\psi\} = \{\psi_1, \psi_2 \ldots \psi_n\}$ we attempt to construct approximate numerical solutions $\tilde{u}$ $(\tilde{u} \approx u)$ taking advantage of the fact that any $\tilde{u}$ belonging to the n dimensional function space spanned by $\{\psi\}$ can be expressed as a linear combination of the basis functions $\psi_j,\ j = 1,2,\ldots n$

$$\tilde{u} = c_1 u_1 + c_2 u_2 + \ldots c_n u_n$$

There are a number of methods which can be used to construct the solution. The methods differ in the choice of the $n =$ dimensional function space spanned by $\{\psi\}$

2. How can least squares minimization be used to solve pdes?

To solve $L\tilde{u} \approx f$ with $\tilde{u} = c_1\psi_1 + c_2\psi_2 + \ldots c_n\psi_n \in H^n$, we have to satisfy : $c_1 L\psi_1 + c_2 L\psi_2 + \ldots c_n L\psi_n$

If least squares minimization in the L_2 norm is used, this requires :

$\left\| f - \sum_{i=1}^{n} c_i L\psi_i \right\|^2 = (f - \sum_{i=1}^{n} c_i L\psi_i, f - \sum_{i=1}^{n} c_i L\psi_i)$ must be a minimum

The coefficients $c_i, i = 1 \ldots n$ are found by solving the system **Ac** = **f** where

$$\mathbf{c} = (c_1, c_2, \ldots c_n)^T \quad \mathbf{f} = \{(L\psi_1, f), (L\psi_2, f) \ldots (L\psi_n, f)\}$$

$$\mathbf{A} = \begin{bmatrix} (L\psi_1, L\psi_1)\ (L\psi_1, L\psi_2) \ldots (L\psi_1, L\psi_n) \\ (L\psi_2, L\psi_1)\ (L\psi_2, L\psi_2) \ldots (L\psi_2, L\psi_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (L\psi_n, L\psi_1)\ (L\psi_n, L\psi_2) \ldots (L\psi_n, L\psi_n) \end{bmatrix}$$

3. What is the collocation method?

If instead of evaluating the inner products $(L\psi_i, L\psi_j)$ and $(L\psi_j, f)$ as integrals over the problem domain, we evaluate the inner product at a discrete set of say 'm' points in the domain, we get the collocation method.

In that case $A_{ji} = \sum_{k=1}^{m} L\psi_j(x_k) L\psi_i(x_k), \quad f_j = \sum_{k=1}^{m} f(x_k) L\psi_j(x_k)$

4. What is the Galerkin method?

Unlike the least squares minimization technique and the collocation method, the Galerkin method determines the coefficients using a different criterion. In the Galerkin method the residual function $f - \sum_{i=1}^{n} c_i L\psi_i$ is required to be orthogonal to each and every basis funciton in H^n. This imposes the criteria :

$$(f - \sum_{i=1}^{n} c_i L\psi_i, \psi_j) = 0 \; \forall \; \psi_j \in H^n, j = 1,2,\ldots n$$

5 What is the best approximation property of Galerkin's method?

In order to understand the best approximation property of Galerkin's method it is necessary to define a new norm and associated inner product in the space of functions H^n. The inner product of functions u and v in the new norm is given by:

$$\langle u,v \rangle = (u, Lv), \quad \langle u,u \rangle = (u, Lu) = \|Lu\|^2 \quad \forall u, v \in H^n$$

This inner product as distinct from the inner product associated with the L_2 norm is defined with respect to the operator L. Given the definition of the $\langle \; \rangle$ norm, the best approximation property of Galerkin's method is states as follows :

If L is positive definite and u^* is the solution of (B) i.e. $Lu^* = f$ on D and $\overline{L}u^* = 0$ on ∂D, then the Galerkin method gives the best possible approximation to u^* in the function space H^n measured in the $\langle , \rangle$ norm.

It can be shown that finding the best possible solution $u \in H^n$ in the $\langle , \rangle$ norm is equivalent to finding the solution using the Galerkin method.

6. What is weighted residual method? Is the Galerkin method a weighted residual method?

The Galerkin method is also an instance of a broader class of methods known as Weighted Residual Methods. In the Weighted Residual Method, the residual $f - \sum_{i=1}^{n} c_i L\psi_i$ is required to be orthogonal to function space H'^n, not necessarily identical to the space H^n, spanned by $\{\psi_j\}, j = 1\ldots n$.

7. What is the finite element method? What is the Bubnov-Galerkin finite element method and the Petrov-Galerkin finite element method?

The Finite element method is a weighted residual method. There are two main variations of the finite element method - the Galerkin finite element method and the Petrov - Galerkin finite element method

In the Galerkin Finite element method, the spaces H^n and H'^n are taken to be identical - thus the Galerkin finite element method is identical to the Galerkin method.

In the Petrov Galerkin finite element method, the function spaces H^n and H'^n are not the same, but the basis functions $\{\psi_j\}$ and $\{\psi'_j\}$, satisfy very similar requirements as in the Galerkin finite element method.

8. What is the weak form? Why is it advantageous for the finite element method to solve the weak form?

In the Finite element method, instead of directly solving $Lu = f$ (the "strong" form of the p.d.e.), an equivalent weak form of the equation ($\hat{L}u = f$) is solved. The operator $\hat{L}$ imposes less stringent continuity requirements on the function u.

This allows H^n to be chosen as a space of basis functions which are low order (e.g.) linear polynomials

9. When is the Galerkin method equivalent to minimizing a functional?

In many problems of mechanics, variational principles exist which lead to derivation of the governing equations of a problem by minimizing a functional: typically the energy functional associated with that problem. The Galerkin method for self adjoint operators L is thus a way to minimize the associated energy functional, if the energy functional can be expressed in the above form, and thus can be regarded as a variational method.