Lesson 34

1. How can orthogonal functions be generated?

Given a linearly independent sequence of functions $\{\varphi_j\}$ it is always possible to construct an orthogonal sequence of functions from them - using the Gramm Schmidt procedure. Orthogonal functions are also generated by the eigen functions of a self adjoint system.

2. Does the "best approximation" always exist and is it unique?

Given a set of orthogonal basis functions, the best approximation for that basis always exists and is unique. The coefficients c_j^* corresponding to the best approximation are given by:

$$c_j^* = \frac{(\phi_j, f)}{(\phi_j, \phi_j)} = \frac{(\phi_j, f)}{\|\phi_j\|^2}$$

which exists since $\|\phi_j\|^2 \neq 0$ if $\phi_j \neq 0$ (since the inner product is positive definite)

3. What are Bessel's inequality and Parseval's formula?

If $f^* = \sum_{j=0}^{n-1} c_j^* \phi_j$ is the best approximation to f, then Bessel's inequality states that

$$\sum_{j=0}^{\infty} (c_j^*)^2 \|\phi_j\|^2 \leq \|f\|^2$$

for an infinite number of basis functions. If the basis functions of the infinite series are bounded then $\sum_{j=0}^{\infty} (c_j^*)^2 \|\phi_j\|^2 = \|f\|^2$. This known as Parseval's formula.

4. What is the general recursion formula for orthogonal polynomials?

For $n \geq 1$ all families of orthogonal polynomials satisfy a three term recursion formula which allows a new member of the family, $\phi_{n+1}(x)$ to be generated from existing members $\phi_n(x)$ and $\phi_{n-1}(x)$. The recursion formula enables $\phi_{n+1}(x)$ to be determined uniquely, up to an arbitrary constant α_n which relates the leading coefficient of $\phi_{n+1}(x), \gamma_{n+1}$ to the leading coefficient of $\phi_n(x)$, γ_n

The recursion formula allows construction of a series of orthogonal polynomials in unique fashion if the first two terms of the series are known.

5. How many zeros does an orthogonal polynomial of degree n have? Are the zeros simple zeros?

By construction the n^{th} order polynomial has n zeros. However, in addition, a n^{th} degree polynomial in a family of orthogonal polynomials with weight function w on an interval $[a,b]$ has n simple zeros, all of which lie in $[a,b]$

6. What are Legendre polynomials?

The Legendre polynomials are a series of orthogonal polynomials which are all roots of Legendre's equation. They are defined by the formula:

$$P_0(x) = 1 \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n], \quad (n = 1,2,\ldots)$$

The inner product, defined over [-1,1], has weight factor 1.

$$\text{Thus}: (P_n, P_j) = \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^j}{dx^j}(x^2-1)^j \, dx = 0 \text{ if } n \neq j.$$

$$= \frac{2}{2n+1} \text{ if } n = j.$$

The Legendre polynomials also satisfy symmetry: $P_n(x) = (-1)^n P_n(x)$ similar to Chebyshev polynomials.