Lesson 33

1. Should Chebyshev interpolation be always used in preference to equidistant interpolation?

   Chebyshev interpolation is probably best for higher order interpolations i.e. $n > 2\sqrt{m}$ where $m+1$ is the number of grid points and n is the order of the polynomial since for $n > 2\sqrt{m}$, equidistant interpolation using polynomials of high degree becomes in some cases an "ill - conditioned" problem. However, finding the grid points of Chebyshev interpolation involves evaluating trigonometric functions. Evaluation of trigonometric functions is computationally expensive : doing so repeatedly for large problems may impose a heavy computational burden. Hence for lower order interpolations, equidistant interpolation may be preferable.

2. When is an interpolant ill-conditioned?

   An interpolant is "ill - conditioned" when the values which one gets by equidistant interpolation with a polynomial of high degree are very sensitive to disturbances in the values of the function. This is particularly true near the ends of the interval.

3. Why are orthogonal polymials useful?

   Orthogonal polynomials form a 'basis' for the ∞ dimensional function space. They are easy to manipulate, have good convergence properties and give a well - conditioned representation of the function : minor changes in function value do not lead to large changes in the values obtained from the interpolation. Because of this functions are often written as expansions in terms of orthogonal polynomials.

4. What is meant by the basis functions being linearly independent?

   The basis functions of a function space $\phi_0, \phi_1, \phi_{2...}$ are linearly independent if

   $\sum_{j=0}^{n-1} c_j \phi_j = 0$ implies all the c_j $(c_0, c_1, \ldots c_{n-1})$ are zero.

   These functions are known as basis functions precisely because of the linear independence property, because any function belonging to the n dimensional space can then be written as a linear combination of these basis functions :

   $\varphi^n = \sum_{j=0}^{n-1} c_j \phi_j$ where φ^n is an arbitrary member of the n dimensional function space

5. <u>What is the inner product between two functions?</u>

In general the inner product between two functions is defined by:

$$(f,g)=\int_a^b f(x)g(x)w(x)\,dx$$

The L_2 norm of a function is expressed in terms of its inner product as:

$$\|f\|_{L_2}^2=\int_a^b f(x)f(x)w(x)\,dx$$

In the discrete case the inner product becomes $\sum_{i=0}^{m} f(x_i)g(x_i)w(x_i)$ where m is the number of grid points at which the function values are known.

6. <u>What are the rules followed by the inner product between two functions?</u>

The inner product of two functions f and g obeys the same rules as the scalar product of two vectors. Thus:

$$(f,g)=(g,f) \qquad \text{(commutativity)}$$

$$(c_1 f+c_2 g,\phi)=c_1(f,\phi)+c_2(g,\phi) \qquad \text{(linearity)}$$

$$(f,f)>0\ \forall\ f\neq 0\ \forall\ x\in[a,b]$$

$$(f,f)=0 \text{ only if } f=0\ \forall\ x=0 \text{ (positive definiteness)}$$

7. <u>When is the representation of a function in terms of orthogonal basis functions the "best approximation" to the function?</u>

The best approximation is the solution to the approximation problem, that in the L_2 norm, is that linear combination of basis functions whose distance from the target function f is minimum.

If we denote this distance as the error vector f^*-f, the magnitude or norm of this error vector is going to be a minimum when f^*-f is perpendicular to the space spanned by $\phi_0,\phi_1,\ldots\phi_{n-1}$. Thus the coefficients $c_0,c_1,\ldots c_{n-1}$ are determined to satisfy the requirement that the squared norm $\left\|\sum_{j=0}^{n-1} c_j\phi_j - f\right\|^2$ be as small as possible i.e. a minimum.

8. <u>What is the Pythagorean theorem for orthogonal functions?</u>

Orthonormal basis functions have several desirable properties. For instance they satisfy the Pythagorean theorem. If f and g are orthogonal i.e. $(f,g)=0$ and $\|f\|\neq 0$ and $\|g\|\neq 0$ then:

$$\|f+g\|^2=(f+g,f+g)=(f,f)+(g,f)+(f,g)+(g,g)=\|f\|^2+\|g\|^2$$