Lesson 32

1. Why are the Chebyshev polynomials orthogonal polynomials?

The Chebyshev polynomials are orthogonal polynomials because they are orthogonal with respect to the weighting function $(1-x^2)^{-\frac{1}{2}}$

Thus $\int_{-1}^{1} T_i(x)T_j(x)(1-x^2)^{-\frac{1}{2}}dx = 0 \quad \text{if } \; i \neq j$

$$= \frac{\pi}{2} \text{ if } \; i = j \neq 0$$

$$= \pi \text{ if } \; i = j = 0$$

2. How many zeros and extrema does a Chebyshev polynomial of order *n* possess?

$T_n(x)$ has n zeros in [-1,1] and the zeros are given by:

$$T_n(x_k) = \cos(n\phi_k) = 0$$

$$\therefore n\phi_k = (2k+1)\frac{\pi}{2} \quad k = 0,1,2,\ldots n$$

$$\phi_k = \frac{(2k+1)}{n}\frac{\pi}{2}, \quad x_k = \cos\phi_k = \cos[\frac{(2k+1)}{n}\frac{\pi}{2}]$$

$T_n(x)$ also has $n+1$ extrema in [-1,1]

3. What are discrete Chebyshev polynomials and how do they satisfy the orthogonality property?

Discrete Chebyshev polynomials are Chebyshev polynomials evaluated at discrete points. They satisfy the orthogonality property in the following manner:

$$\sum_{k=0}^{m} T_i(x_k)T_j(x_k) = 0 \text{ if } \; i \neq j$$

$$= \frac{1}{2}(m+1) \text{ if } \; i = j \neq 0$$

$$= (m+1) \text{ if } \; i = j = 0$$

where the x_k's are the $(m+1)$ zeros of the Chebyshev polynomial of order $(m+1)$

4. What is the smallest norm property of Chebyshev polynomials?

A very important property of Chebyshev polynomials is the property that among all polynomials of order $n, 0 < n < \infty$, which have leading coefficient of one, $2^{1-n}T_n$ i.e. the Chebyshev polynomial of order n scaled by 2^{1-n} has the smallest maximum norm in the interval [-1,1].

By dividing each term of any polynomial of order n we can ensure that the leading order term in the polynomial has coefficient one. Thus $2^{1-n}T_n$ has the lowest maximum norm in [-1,1] for the entire set of polynomials of order n, suitably scaled.

5. <u>Why is the smallest norm property of Chebyshev polynomials useful?</u>

The smallest norm property of Chebyshev polynomials is useful in determining the optimum location of the grid points. Because of this property it is guaranteed that if a m^{th} order polynomial is chosen to approximate a function then the error would have the minimum possible value in the interval [-1,1] and thus by extension [a, b] if the grid points are chosen to be the zeros of T^{m+1}. Interpolations that use these grid points are known as Chebyshev interpolations