Lesson 30

1. How is the Laplacian operator for a regular triangular grid formed?

For a regular triangular grid, the Laplacian at an arbitrary point ***x*** located within a triangle is the weighted sum of the five point Laplacian operators calculated at the three vertices of the triangle with the weights being the barycentric coordinates of the point ***x.***

2. How is the Laplacian operator in an irregular triangular grid calculated?

The irregularity of the grid results in a modification to the expression for the Laplacian difference operator. The modified expression includes the expression for the five point Laplacian difference operator on a regular triangular grid together with some additional terms. These correction terms represent contributions from mixed derivatives i.e numerical approximations to $\frac{\partial^2 u}{\partial x \partial y}$.

3. How is the error in the approximation of a function f(***x***) by a n^{th} order polynomial defined?

Suppose a function f is approximated by a polynomial $p_n(x)$ of order n in the interval [a, b]. Let us assume that the function f is continuous in [a, b]. Then $\|f - p_n(x)\|_\infty$ is the norm of the error in the infinite norm, evaluated point wise in [a, b] due to the approximation of f by $p_n(x)$ in [a, b]. The theoretical bound (minimum) value of this error for a n^{th} order polynomial is denoted $E_n(f)$.
It is obtained by varying the polynomial over the set of all the n^{th} order polynomials and evaluating point wise the error using the infinite norm.
$E_n(f) = \min_{p_n(x) \in P_n(x)} [\max_{x \in [a,b]} \|f(x) - p_n(x)\|]$ where $P_n(x)$ is the set of all n^{th} order polynomials.

4. What is the Weirstrass approximation theorem?

The Weirstrass Approximation theorem states that as $n \to \infty$, $E_n(f) \to 0$.
What this means is that at least theoretically, by increasing the order of polynomial without limit it is possible to obtain a polynomial of sufficiently high order such that for that polynomial and for polynomials of higher order, the error in the infinite norm becomes zero.

5. How does the smoothness of the unknown function or the size of the interval over which the function is being approximated influence the behaviour of the error?

It is seen that the smoother the function $f(x)$ the faster the reduction in $E_n(f)$ with polynomial order. Also the size of the interval is crucial, the smaller the interval $[a,b]$ smaller the value of $E_n(f)$ and faster the reduction of $E_n(f)$ with poynomial size over the interval. This means that over a narrow interval a function may be approximated with considerable accuracy by a relatively lower order polynomial.

6. How does the Weirstrass approximation theorem provide a justification for the quadratic convergence of Newton's method near the root?

Near a root of any function, the Newton Raphson method that near behaves "like a quadratic", and since the Newton Raphson slope exactly matches the slope of a quadratic passing through the point, the Newton Raphson method converges quadratically near the root for any non - linear function. The Weirstrass approximation theorem provides a justification for the claim that any function behaves "like a quadratic" in a sufficiently small interval near the root since for a sufficiently small interval $E_2(f) \approx 0$ as per the Weirstrass approximation theorem. For large intervals $[a,b]$ on the other hand, $E_n(f)$ decreases so slowly with increasing polynomial order that it becomes infeasible to approximate f with only one polynomial in $[a,b]$

7. Why is it hard to construct the optimum polynomial?

Given a certain interval $[a,b]$ and a polynomial of order n finding the optimal choice of polynomial which minimizes $E_n(f)$ is not easy. The most common methods of constructing polynomials give errors which are significantly larger than $E_n(f)$, the error obtained by an optimal choice of polynomial of order n. If the polynomial constructed is significantly different from the optimal choice of polynomial, there is no guarantee that just by increasing the order of the polynomial the error in the infinite norm will reduce to zero.

8. What is Runge's phenomenon?

It is seen that if the choice of polynomial is not optimal, as the order of the approximating polynomial is increased, then for very high order polynomials (>10), the error in the infinite norm near the boundaries of the interval become very large while the error near the central portion of the interval is small. This is known as Runge's phenomenon.

9. Why is the actual representation of the polynomial important?

Identical choice of polynomial type and order may result in large or small numerical error, depending on the manner in which the polynomial is represented.

For instance, mathematically identical representations of the same polynomial, e.g.

$$p_n(x) = \sum_{k=0}^{n} a_n x^k \quad \text{or} \quad p_n(x) = \sum_{k=0}^{n} a_n (x - x_0)^k$$

can result in different round of errors.

For instance if the polynomial is of interest in the interval [a, b] and if the second representation is used, then if x_0 is chosen to be the mid point of the interval $(a+b)/2$ then the accuracy of the computations improve significantly. Thus the second representation is more desirable in this case.

10. What is the bound on the interpolation error?

If the actual function $f(x)$ which is being approximated $p_m(x)$ has derivatives that are continuous upto order at least $m+1$, i.e. the function is smooth upto order m, then it can be shown that the interpolation error satisfies the following bound :

$$R(x) = f(x) - p_m(x) = \frac{f^{m+1}(\xi)}{(m+1)!}(x - x_0)(x - x_1)\ldots(x - x_m)$$

where ξ is a point in the smallest interval that contains x as well as $x_0, x_1, \ldots x_m$ and is denoted by $\text{int}(x, x_0, x_1, \ldots x_m)$

11. What is Newton's interpolation formula?

The problem of determining a polynomial Q of degree m through $m+1$ points has a unique solution. Newton's interpolation formula is a systematic procedure for obtaining this unique polynomial.