Lesson 29

1. What are multi-dimensional difference operators and why are they required?

   Partial differential equations in multi-dimensions involve partial derivatives with respect to more than one independent variable. Multi-dimensional difference operators are numerical approximations to partial derivatives involving more than one independent variable. Multi-dimensional difference operators are typically implemented in a rectangular grid in 2D and a grid in the form of a rectangular parallelepiped in 3D.

2. In a 2D grid with grid size h_x and h_y what is it that the central difference operator and why is it 2nd order accurate?

   The central difference approximations to the partial derivatives in x and y, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are given by:

   $$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h_x} + error\ terms, \frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} + error\ terms$$

   where a step size that is twice the usual step size for the central difference operator (usually $\delta u = u_{i+1/2} - u_{i-1/2}$) has been used.

   Since for the central difference operator, the following relation holds between the differential and difference operator of order k,

   $$f^{(k)}(a) = h^{-k}\delta^k f(a) + c_1 h^2 f^{k+2}(a) + c_2 h^4 f^{k+4}(a) + \ldots$$

   $\frac{\partial u}{\partial x} = \frac{1}{2h}\delta u + o(h^2)$ and hence the leading term in the error for the central difference approximation to the the first order partial derivatives in x and y is of 2nd order in h

3. In a 2D grid with grid size h_x and h_y what is it that the forward difference operator and why is it 1st order accurate?

   If a forward difference operator is used to approximate the derivative, then we have: $\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h} + error\ terms$. In this case the difference approximation becomes unsymmetric. In addition, the error is of first order, not second orde in the step size.

The reason for the reduction in accuracy is because the forward difference operator and the first order derivative are related by :

$$\Delta^k f(x) = h^k f^{(k)}(\xi) \text{ where } \xi \in [x, x+kh]$$

Since $\xi \to x$ linearly with step size h, the error too reduces linearly with step size. Thus $\frac{\partial u}{\partial x} \approx \frac{\Delta u}{h} + o(h)$

4. <u>What is the simplest approximation to the Laplacian operator in 2D?</u>

The simplest approximation to the Laplacian operator is constructed using the 2nd order difference operator in two dimensions.

The 2nd order central difference operator is given by :

$$\delta^2 f(a) = f(a+h) - 2f(a) + f(a-h)$$

Also, since $f^2(a) = h^{-2}\delta^2 f(a) + c_1 h^2 f^4(a) + c_2 h^4 f^6(a)$ for the central difference operator, the error in the approximation is of 2^{nd} order. Using the central difference approximation to the second order partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ one can construct the five point central difference approximation to the Laplacian : $\nabla_5^2 u_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2}$

5. <u>What is the truncation error in the five point central difference approximation to the Laplacian?</u>

The order of the error in the five point operator approximatingthe Laplacian is $o(h^2)$ since :

$$\nabla_5^2 u_{ij} = \nabla^2 u + \frac{1}{12} h^2 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \ldots.$$