Lesson 28

1. What are the different types of difference operators?

   The central difference operator, denoted by δ operates on $f(x)$ to yield $f(x+\frac{1}{2}h)-f(x-\frac{1}{2}h)$.

   The average difference operator μ operating on $f(x)$ yields,

   $\mu f(x)=\frac{1}{2}[f(x+\frac{1}{2}h)+f(x-\frac{1}{2}h)]$

   The backward difference operator is defined by:

   $\nabla f(x)=f(x)-f(x-h)$

2. Why is it that for functions that are 2nd or 3rd order polynomials, the second order central difference operator is the closest numerical difference approximation to the exact second derivative?

   Similar to the relationship between the k^{th} order forward difference operator and the k^{th} order derivative i.e. between $\Delta^k f(x)$ and $f^{(k)}(x)$, there exists a relationship the k^{th} order central difference operator between $\delta^k f(x)$ and $f^{(k)}(x)$:

   $$f^{(k)}(a)=h^{-k}\delta^k f(a)+c_1h^2f^{(k+2)}(a)+c_2h^4f^{(k+4)}(a)+\ldots$$

   The right hand side includes only even derivatives of f. Also, the constants c_l are independent of f but depend on k.

   Hence, for 2nd-degree or 3rd degree polynomials,

   $$f''(a)=\frac{\delta^2 f(a)}{h^2}=\frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

   Thus in this case theoretically the second order central difference operator is exactly equal to the second derivative of the function irrespective of the step size. However in practical applications the computations will involve roundoff errors. Roundoff errors will have significant effect on the accuracy of the higher differences and they will not represent the exact derivatives.

3. Why may it be necessary to extrapolate function values in order to compute difference operators?

It is apparent that using the difference operators to estimate the derivatives of a function requires knowledge of the function values at various points on a grid $(x_0, x_1 \ldots x_n)$. Often knowledge of the function and its derivaties is incomplete i.e. the function values and possibly some of the derivatives are known at some points on the grid but not at all points in the grid. One way to do this is to fit a poly-nomial of the highest possible order through the known function values and use this polynomial to estimate the value of the function at the unknown grid points. The truncation error due to this estimation procedure reduces with grid size. However, the work to estimate the function values at unknown grid points sharply increases as the grid size becomes smaller. In addition round off errors increase as the grid size reduces.

4. What is Richardson extrapolation and why is it useful?

The Richardson extrapolation technique enables calculation of the limiting value of a quanitity as the grid size approaches zero, without actually having to reduce the grid size to zero. Thus if it is necessary to estimate the function value at an unknown grid point it is no longer necessary to reduce the grid size in order to reduce the truncation error in the estimate.

5. What is operator calculus?

The difference operators, *E, Δ, δ, μ* etc. can be operated on using the rules of calculus to find approximation formulas. However this requires making assumptions on the continuity and differentiability of the space of functions on which these operators are defined. Operator calculus can be used to establish relations between the various difference operators.