

Lesson 27

1. How are differential operators represented numerically?

The first step in the numerical solution of partial differential equations is to represent the differential operators appearing in the equations by appropriate numerical representations. These numerical representations are known as difference operators.

2. Is there an error introduced due to the approximation of differential operators by numerical operators?

Representation of differential operators by numerical operators leads to errors – both truncation as well as round off errors. Accurate numerical analysis of pde's requires awareness of the magnitude of the error introduced because of the numerical representation of the differential operators. This requires knowledge of the leading order term in the error and how it depends on the fineness of the discretization of the spatial and/or time domain.

3. What is the shift operator?

Let y represent a sequence $\{y_0, y_1 \dots y_n\}$. The shifting operation E operates on each term of the sequence and advances each term by 1 to the right to create a new sequence.

Thus $Ey = \{y_1, y_2, y_3, \dots y_{n+1}\}$. Repeated application of the shift operator to a sequence $\{y\}$ leads to additional sequences e.g. $E^k y = \{y_k, y_{k+1} \dots\}$

The shift operator can be applied fruitfully to functions as well as sequences.

E relates a function f , evaluated at a certain value of its argument, say x , to the function value evaluated at $x+h$, where h is the step size

$$Ef(x) = f(x+h)$$

4. What is the forward difference operator?

The forward difference operator Δ operates on the sequence $\{y_0, y_1 \dots y_n\}$ to create a new sequence by subtracting each term in the sequence from the term to its immediate right. Hence, $\Delta y = \{y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots y_{n+1} - y_n\}$

Repeated application of the forward difference operator leads to the additional sequences.

For example $\Delta^k y$, known as the k th difference of the sequence y is a sequence where each term in the sequence involves $k+1$ terms of the original sequence.

This leads to rapid growth in the number of terms e.g.

$$\Delta^2 y = \Delta\{y_{n+1} - y_n\} = \{(y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)\} = \{y_{n+2} - 2y_{n+1} + y_n\}$$

The difference operations can be applied fruitfully to functions as well as sequences. Δ applied to a function, relates a function f , evaluated at a certain value of its argument, say x , to the function value evaluated at $x+h$, where h is the step size

$$\Delta f(x) = f(x+h) - f(x)$$

Unlike sequences however, the values of the difference operators when applied to functions, depends on the step size.

5. Why is the error in the derivative, approximated in terms of difference operators, always higher than the error in the initial data?

If the sequence $y = \{0,0,0,0,0\}$ is changed to the sequence $y = \{0,0,0,1,0\}$ just by perturbing one element of the sequence, then repeated applications of the difference operator to the sequence shows that the difference broadens out and grows quickly. This leads to the conclusion that the error in the derivative is always higher than the error in the initial data

In case of a known function, evaluation of successive higher order differences leads to the conclusion that differences decrease rapidly as the order of the differences increase. When the differences become small enough round off error dominates: thus higher order differences may not contain any meaningful information.

6. How are the differential and difference operators related?

Differentiation is a limiting case of forming differences by applying difference operators. As the step size h approaches zero, the results of difference operations approach the results of differentiation.

In general, it can be shown that $\Delta^k f(x) = h^k f^{(k)}(\xi) \quad \xi \in [x, x+kh]$

Thus $h^{-k} \Delta^k f(x)$ is an approximation to $f^{(k)}(x)$. The error in the approximation approaches zero as $h \rightarrow 0$ and the error is approximately proportional to h (error is linear in h)