

## Lesson 26

1. Can Green's third identity be directly used to find a solution to Laplace's equation?

Green's third identity suggests that if we know  $\phi$  as well as its normal derivatives at the boundary it is possible to determine  $\phi$  throughout the domain using this result. However in practice either the derivative or the function are known at the boundary, specifying both is likely to make the problem ill-posed. This prevents Green's third identity to be used directly to find the solution of Laplace's equation.

2. If  $\phi$  is a solution to Laplace's equation, why is  $\nabla^n \phi, n = 1 \dots \infty$  also a solution?

Since Laplace's equation is a linear equation and the coefficients are constant, if  $\hat{\phi}$  is a solution, i.e.  $\nabla^2 \hat{\phi} = 0$ , then  $\nabla^2 \nabla \hat{\phi} = 0$ . Hence  $\nabla \hat{\phi}$  as well as  $\nabla^n \hat{\phi} \ n = 1 \dots$  are solutions of Laplace's equation.

3. What is Green's function? Write down the solution of Laplace's equation in terms of Green's function?

In Green's second identity if we write  $\psi = G = \frac{1}{4\pi R} + U$  where  $R = |\mathbf{x} - \mathbf{y}|$  and

$U$  is harmonic in  $\mathbf{y}$  i.e.  $\nabla_{\mathbf{y}}^2 U = 0$  in domain  $V$ , we get :

$$\int_V (G \nabla^2 \phi - \phi \nabla^2 G) d\mathbf{y} = \int_{\partial V} (G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}) dS_y$$

But from the fundamental solution,  $\nabla^2 (-\frac{1}{4\pi R}) = \delta(\mathbf{x} - \mathbf{y})$ . Hence :

$$\int_V G \nabla^2 \phi d\mathbf{y} + \phi(\mathbf{x}) = \int_{\partial V} (G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}) dS_y$$

Therefore,

$$\phi(\mathbf{x}) = \int_{\partial V} (G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n}) dS_y - \int_V G \nabla^2 \phi d\mathbf{y}$$

The function  $G$  in the above equation is a Green's function.

4. How can the Green's function be used to find solutions of Laplace's equation?

If we can find  $G$  such that  $G = 0$  on  $\partial V$ , then in case  $\nabla^2 \phi = 0$ , we can find  $\phi$

from the above:  $\phi(\mathbf{x}) = - \int_{\partial V} \phi \frac{\partial G}{\partial n} dS_y$ . The only requirement is that  $\phi$  must have

Dirichlet boundary conditions prescribed on  $\partial V$

Similarly if we can find  $G$  such that  $\frac{\partial G}{\partial n} = 0$  on  $\partial V$ , then solutions for  $\nabla^2 \phi = 0$  can

be found using:  $\phi(\mathbf{x}) = \int_{\partial V} G \frac{\partial \phi}{\partial n} dS_y$ . In this case,  $\frac{\partial \phi}{\partial n}$  must be known on  $\partial V$  i.e.  $\phi$

must have Neumann boundary conditions prescribed on  $\partial V$ .

5. What are spherical harmonics and why are they useful in finding solutions to Laplace's equation?

In certain physical problems, the solution to Laplace's equation is best expressed in terms of a series solution. For example, for the case of a spherical inclusion moving with a velocity  $u$  in an infinite fluid medium which is otherwise motionless, and the fluid is incompressible as well as irrotational, the resultant motion of the fluid can be found by solving Laplace's equation.

The final expression for  $\phi(\mathbf{x})$  can then be expressed as a series solution :

$$\phi(\mathbf{x}) = A^{(0)} \nabla_{\mathbf{x}}^{(0)} \left(\frac{1}{r}\right) + \mathbf{A}^{(1)} \nabla_{\mathbf{x}} \left(\frac{1}{r}\right) + \mathbf{A}^{(2)} \nabla_{\mathbf{x}}^2 \left(\frac{1}{r}\right) + \dots = \sum_{n=1}^{\infty} \mathbf{A}^{(n)} (\nabla_{\mathbf{x}})^n \frac{1}{r}$$

$A^{(0)}$  is a scalar,  $\mathbf{A}^{(1)}$  is a vector with components  $\{A_1^{(1)}, A_2^{(1)}, A_3^{(1)}\}$  while  $\mathbf{A}^{(2)}$  has components  $\{A_{11}^{(2)}, A_{12}^{(2)}, A_{13}^{(2)}, A_{21}^{(2)}, \dots, A_{33}^{(2)}\}$ . Each term in the series is a solution of Laplace's equation.

The set of solutions to Laplace's equation,  $\nabla^n \left(\frac{1}{r}\right), n = 0, 1, \dots, 2$  are

known as spherical harmonics

6. What is Legendre's equation and when does it arise?

The solution to Laplace's equation in a spherical domain can also be obtained by seeking a solution in the separated form:

$$\phi = G(r)H(\tilde{\phi})I(\theta)$$

This gives rise to Legendre's equation and yields a series expansion in terms of spherical harmonics that involves Legendre's polynomials.