Lesson 25

1. Why is the Laplacian operator translationally invariant in space?

Translational invariance means that if we have ϕ satisfying Laplace's equation in the x_1,x_2,x_3 system as $\nabla^2_{\mathbf{x}}\phi = \frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}=0$ and the origin of the x_1,x_2,x_3 is shifted to (x_1^0, x_2^0, x_3^0) such that we get transformed coordinates like $x_1' = x_1 - x_1^0$, $x_2' = x_2 - x_2^0, x_3' = x_3 - x_3^0$ then :

$$\nabla^2_{\mathbf{x}'} = \frac{\partial^2\phi}{\partial x_1'^2}+\frac{\partial^2\phi}{\partial x_2'^2}+\frac{\partial^2\phi}{\partial x_3'^2}=0$$

2. What is the basic form of the fundamental solution to Laplace's equation?

The solution of the $\nabla^2_{\mathbf{x}}\hat{\phi}(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, obained as :

$$\hat{\phi} = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} = \frac{-1}{4\pi\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}$$

is known as the basic form of the fundamental solution to Laplace's equation.

3. What is Poisson's equation?

Poisson's equation is Laplace's equation with a non-zero right hand side:

$$\nabla^2_{\mathbf{x}}\bar{\phi} = \rho(\mathbf{x})$$

4. What is the Green's function approach to the solution of Laplace's equation?

It is possible to construct solutions to Laplace's equation using Green's functions. This is an extremely powerful approach, since they allow solutions to Laplace's equation to be obtained for general boundary conditions.

Greens functions are constructed to satisfy Laplace's equation as well as boundary conditions homogeneously i.e. irrespective of the actual boundary conditions the Green's function is always zero at the boundary.

To construct Green's function we make use of Green's identities.

5. What is Green's first identity?

If we consider two smooth functions ϕ and ψ defined over a volume V with boundary ∂V, then we can write Green's first identity as :

$$\int_V \psi \nabla^2 \phi d\mathbf{y} = \int_V \nabla.(\psi \nabla \phi) d\mathbf{y} - \int_V \nabla \psi . \nabla \phi \, d\mathbf{y} =$$

$$\int_{\partial V} \psi \nabla \phi . \mathbf{n} d\mathbf{y} - \int_V \nabla \psi . \nabla \phi \, d\mathbf{y} = \int_{\partial V} \psi \frac{\partial \phi}{\partial n} dS_{\mathrm{y}} - \int_V \nabla \psi . \nabla \phi \, d\mathbf{y}$$

6. <u>What is Green's second identity?</u>

If we interchange ϕ and ψ in Green's first identity, we get:

$$\int_V \phi \nabla^2 \psi \, d\mathbf{y} = \int_{\partial V} \phi \frac{\partial \psi}{\partial n} dS_{\mathrm{y}} - \int_V \nabla \phi . \nabla \psi \, d\mathbf{y}$$

Subtracting the above expression from Green's first identity we get Green's second identity:

$$\int_V \psi \nabla^2 \phi - \phi \nabla^2 \psi \, d\mathbf{y} = \int_{\partial V} (\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}) dS_{\mathrm{y}} = 0$$

7. <u>What is Green's third identity?</u>

To get Green's third identity we assume ϕ is harmonic, i.e. $\nabla^2 \phi = 0$. In addition we take ψ to be the fundamental solution i.e. $\psi = -\hat{\phi} = \frac{1}{4\pi r}$

Then substituting in Green's second identity, we get:

$$-\int_V \phi(\mathbf{y}) \nabla^2 \frac{1}{4\pi r} d\mathbf{y} = \int_{\partial V} [\frac{1}{4\pi r} \frac{\partial \phi}{\partial n} - \phi(\mathbf{y}) \frac{\partial}{\partial n} (\frac{1}{4\pi r})] dS_{\mathrm{y}}$$

Recalling that $\int_V \nabla^2_{\mathbf{x}} (-\frac{1}{4\pi r}) \phi(\mathbf{y}) d\mathbf{y} = \int_V \delta(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y}) d\mathbf{y} = \phi(\mathbf{x})$

Hence $\varphi(x) = \frac{1}{4\pi} \int_{\partial V} [\frac{1}{r} \frac{\partial \phi(\mathbf{y})}{\partial n} - \phi(\mathbf{y}) \frac{\partial}{\partial n} (\frac{1}{|\mathbf{x} - \mathbf{y}|})] dS_{\mathrm{y}}$ which is Green's third identity.