

## Lesson 24

1. Describe two civil engineering applications that are described by Laplace's equation?

A soap film stretched across a loop of wire behaves like a membrane with edge support under transverse loads. The solution of this problem is also used to find the torsional shear stress distribution in solid sections with arbitrary cross-sections. If the displacement at every point in the soap film can be found, the shear stress distribution in the cross section can be calculated as well. The deflection of the thin soap film is obtained by solving Laplace's equation.

The flow of an incompressible, irrotational fluid also satisfies Laplace's equation.

2. What are Dirichlet and Neumann boundary conditions?

Two main types of boundary value problem are associated with Laplace's equation - Dirichlet and Neumann boundary conditions. In the Dirichlet problem,  $\phi$  is given on the boundary i.e. we solve  $\nabla^2 \phi = 0$  subject to  $\phi = c$  on  $\partial B$ . In the Neumann problem we solve  $\nabla^2 \phi = 0$  subject to specification of  $\frac{\partial \phi}{\partial \mathbf{n}} = 0$  on  $\partial B$

3. What is the mean value theorem for harmonic functions?

A function  $\phi(\mathbf{x})$  which satisfies Laplace's equation  $\nabla^2 \phi = 0$  is said to be a harmonic function. Harmonic functions possess certain very useful properties e.g. they satisfy the Mean Value Theorem. According to the Mean Value Theorem a harmonic function  $\phi(\mathbf{x})$  at a point  $\mathbf{x}_0$  is equal to the mean value of  $\phi$  over any sphere with center  $\mathbf{x}_0$  in the domain of harmonicity.

4. What is the fundamental solution of Laplace's equation?

The Laplacian operator in spherical coordinates is expressed in terms of the variables  $r, \theta, \psi$  as:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \psi^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)$$

Considering a function  $\hat{\phi} = -\frac{1}{4\pi r}$ , it is clear that if  $r \neq 0, \nabla^2 \hat{\phi} = 0$ . This solution of Laplace's equation in spherical coordinates is known as the fundamental solution.