Lesson 22

1. What are periodic boundary conditions?

   Periodic boundary conditions ensure that the unknown variable and possibly its derivatives have the same values on entering and leaving the problem domain. In the one-dimensional case, with domain length h, and unknown variable $u(x)$ this implies that $u(x = 0) = u(x = h)$.

2. How can determining the eigen functions help in determining the solution of a partial differential equation?

Since the eigen functions $u_n(x)$ form a basis, any solution to the partial differential equation $Lu = \frac{\partial u}{\partial \mathrm{t}}$ may be written as $u = \sum_{n=0}^{\infty} a_n(t)u_n(x)$ and since the $u_n(x)$ are orthogonal to each other, $a_n(t) = (u_n(x), u)$. We can get initial conditions for $a_n(t)$ from the initial conditions for u: $f(x) = u(0) = \sum_{n=0}^{\infty} a_n(0)u_n(x)$. Hence, $a_n(0) = (u_n(x), f)$.

To find a general solution for $a_n(t)$, take inner product of $\frac{\partial u}{\partial \mathrm{t}} = Lu$ with $u_n(x)$:

$$(u_n(x), \frac{\partial u}{\partial t}) = (u_n(x), \frac{\partial}{\partial t}\sum_{k=0}^{\infty} a_k(t)u_k) = (u_n(x), \sum_{k=0}^{\infty} \frac{\partial a_k(t)}{\partial t} u_k) = \frac{\partial a_n(t)}{\partial t}$$

But $(u_n, \frac{\partial u}{\partial t}) = (u_n, Lu) = (Lu_n, u) = (-\lambda_n u_n, \sum_{k=0}^{\infty} a_k(t)u_k) = -\lambda_n a_n$, using the fact that L is self-adjoint. Then the a_n can be found from solutions of the first order differential equation: $\frac{\partial a_n}{\partial t} = -\lambda_n a_n$.

This solution is of the form: $a_n = Ae^{-\lambda_n t}$ where A is a constant of integration.

Using the initial condition: $a_n(0) = (u_n, f)$ it is clear that $A = (u_n, f)$

Hence $a_n = (u_n, f)e^{-\lambda_n t}$ and hence the solution to (**) is of the form:

$$u = \sum_{n=0}^{\infty} e^{-\lambda_n t}(u_n, f)u_n$$

Hence the entire solution can be constructed from the eigen functions and eigen values.

3. What is the method of images?

   The method of images is used to extend the solution for the wave equation found over a domain with periodic boundary conditions to a finite interval i.e. a domain where the boundaries are not periodic.

4. What is the method of characterisitics?

It is a geometrical method to find solution to the wave equation over a certain domain by making use D'Alembert's general solution to the wave equation.

5. What is the domain of dependence in the method of characteristics?

The solution to the wave equation at any point (x_0, t_0) has a domain of dependence on the initial conditions that lie on the $t = 0$ line. The solution at (x_0, t_0) depends on the value of the initial conditions in the interval $[x_0 - ct_0, x_0 + ct_0]$. Thus the domain of dependence of the solution at (x_0, t_0) not only lies on the line t = 0 but encompasses the entire space time domain enclosed by the characteristic lines passing through the point (x_0, t_0).

6. What is the domain of influence in the method of characteristics?

Similar to the domain of dependence, points on the initial line, say $t=0$, have a domain of influence comprising the part of the space time domain where the solution is affected by the value of f and g at that point.

Characteristic lines emanating from a point x_0 on the $t=0$ line will affect the solution at all points along their length.These points in turn have their own characteristic lines, and once the information from x_0 reaches them, say at $t^*>0$, they will start emitting information along their own characteristic lines that will affect the solution lying at all points on those lines. All these points together will comprise the domain of influence.