

## Lesson 21

### 1. What are initial and boundary conditions?

Initial conditions give information on the state of the unknown quantity (the dependent variable) or its time derivatives at the start of the analysis. Boundary conditions, on the other hand, provide information on the value of the unknown quantity, or its spatial derivatives, at the spatial boundaries of the domain over which the partial differential equation is being solved.

### 2. Why are initial and boundary conditions necessary?

Second order partial differential equations in two variables, for instance, after integration, give rise to four integration constants. To evaluate these four constants, boundary/initial conditions are required. If the independent variables are all spatial variables, these conditions are known as boundary conditions. If one of the independent variable is time, then at least two of these conditions would have to be initial conditions. For example, for the one dimensional wave equation,  $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ , where one of the independent variables is a spatial variable, and the other independent variable is time, two boundary conditions and two initial conditions are required.

### 3. What is the general form of the boundary conditions?

The general boundary condition may involve either the function 'u' or its spatial derivative or a combination of both. They are of the form:

$$\alpha_i u(a) + \beta_i u'(a) = 0$$

where  $i = 1, \dots, n$  and  $n$  is the order of the highest spatial derivative appearing in the partial differential equation.

### 4. What is a linear differential operator?

We consider a general form of the partial differential equation  $Lu = \frac{\partial u}{\partial t}$  with initial condition  $u(t = 0) = f$  and homogeneous boundary conditions.  $L$  operating on  $u$  gives the partial derivatives of  $u$  with respect to the independent spatial variables.

The differential operator  $L$  is linear when it satisfies the following conditions:

$$L(u + v) = L(u) + L(v)$$

$$L(\alpha u) = \alpha L(u) \text{ where } \alpha \text{ is an arbitrary scalar}$$

6. What is the form of the eigen problem for a linear differential operator?

If the eigen functions of  $L$  are denoted by  $u_n$  then under the same boundary conditions as the original problem  $Lu = \frac{\partial u}{\partial t}$ ,  $L$  must satisfy the eigenvalue problem stated as :

$$Lu_n = -\lambda u_n$$

7. What is a self-adjoint linear differential operator? What properties do the eigen functions of linear self-adjoint operator possess?

A linear differential operator is self - adjoint when it satisfies the following condition :  $(Lu, v) = (u, Lv)$  where  $(, )$  is the inner product in the function space to which both functions  $u(\mathbf{x}, t)$  and  $v(\mathbf{x}, t)$  belong. In analogy with matrices, one can say that  $L$  is 'symmetric'

Since  $L$  is self - adjoint, the eigenfunctions, which are solutions to the eigenvalue problem for  $L$  form an orthonormal set  $\{u_n\}$  which is complete. A complete set of orthonormal eigen functions forms a basis for the infinite - dimensional function space

This is very similar to the eigen vectors of a symmetric matrix of dimension  $n \times n$  forming the basis for the  $n$  dimensional vector space.