

Lesson 20

1. What is the most general form of the linear second order Partial Differential Equation (PDE) in two variables?

$$a\varphi_{x_1x_2} + 2b\varphi_{x_1x_2} + c\varphi_{x_2x_2} + 2d\varphi_{x_1} + 2e\varphi_{x_2} + f\varphi = 0$$

Here φ is the physical quantity of interest whose functional dependence on the two independent variables x_1 and x_2 are sought to be found. a, b, \dots, f are constants with no dependence on the independent variables x_1 and x_2 – otherwise the relationship would be nonlinear. The same form of the equation is retained if the number of independent variables is greater than two.

2. What is a hyperbolic linear second order PDE in two variables?

When $a=1, b=d=e=f=0$ and $c = a$ constant the general form specializes to

$$\varphi_{x_1x_2} + c\varphi_{x_2x_2} = 0$$

which is the general form of a hyperbolic linear PDE in two variables. When x_1 is a spatial variable and x_2 is equal to the time variable and $c = -\frac{1}{c^2}$ where c is the speed of a travelling wave, we get the one-dimensional wave equation, probably the simplest hyperbolic linear second order partial differential equation.

3. What is a parabolic linear second order PDE in two variables?

When $a = a$ constant, $b = c = d = e = f = 0$ and $e = -\frac{1}{2}$ the general form specializes to

$$a\varphi_{x_1x_2} = \varphi_{x_2}$$

which is the general form of a parabolic linear PDE in two variables. When x_1 is a spatial variable and x_2 is equal to the time variable and $a = K$ where K is the diffusivity in a medium, we get the one-dimensional diffusion equation, probably the simplest parabolic linear second order partial differential equation.

4. What is an elliptical linear second order PDE in two variables?

When $a = c = 1, b = d = e = f = 0$ the general form specializes to

$$\varphi_{x_1x_2} + \varphi_{x_2x_2} = 0$$

which is the general form of a elliptic linear PDE in two variables. When both x_1 and x_2 are spatial variables a, we get the two-dimensional Laplace's equation, probably the simplest elliptic linear second order partial differential equation.

5. What is the canonical form of the second order linear partial differential equation?

The canonical forms can be obtained by transforming linearly the variables x_1 and x_2 in a manner that results in the coefficients of the mixed partial derivatives as well as the coefficients of the first order partial derivatives becoming zero.

6. What are the canonical forms of the hyperbolic, parabolic and elliptic second order linear partial differential equations?

The canonical form of the hyperbolic equation is given by:

$$\left[\frac{\partial^2 \Phi}{\partial \alpha^2} - \frac{\partial^2 \Phi}{\partial \beta^2} \right] + K\Phi = 0$$

The canonical form of the parabolic equation is given by:

$$\frac{\partial^2 \Phi}{\partial \rho^2} = D \frac{\partial \Phi}{\partial \eta}$$

The canonical form of the elliptic equation is given by:

$$\left[\frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial^2 \Phi}{\partial \rho^2} \right] + K\Phi = 0$$