Lesson 14

1. What is a system of nonlinear equations?

   A system of nonlinear equations in n unknowns x_1, x_2, x_3…..x_n is a series of n equations:

   $$f_i(x_1, x_2, \ldots x_n) = 0, \quad i = 1,2,\ldots n$$

   A one point iteration method for this equation can be constructed by rewriting the system as:

   $$x_i = \phi_i(x_1, x_2, \ldots x_n), \; i = 1,2\ldots n$$

2. Can a system of nonlinear equations be solved iteratively?

   A system of nonlinear equations can be solved iteratively. For example the one point scheme described above can be solved iteratively as:

   $$x_i^{(k+1)} = \phi_i(x_1^{(k)}, x_2^{(k)}, \ldots x_n^{(k)}), \; i = 1,2\ldots n$$

   This can be restated in vector notation as:

   $$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n), \quad \boldsymbol{\varphi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots \phi_n(\mathbf{x}))$$

   $$\mathbf{x}^{(k+1)} = \boldsymbol{\varphi}(\mathbf{x}^{(k)}), \quad k = 0,1,2,\ldots n$$

3. What is the convergence criteria for systems of nonlinear equations?

   In order to define the convergence criteria for a system of nonlinear equations it is necessary to consider the Jacobian of the system. For example, for the one point iterative scheme, the Jacobian is given by:

   $$\mathbf{J} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots\cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots\cdots & \frac{\partial \phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots\cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix}$$

   Each of the partial derivatives must exist in the region $x \in \Re$ where $\Re = \{\mathbf{x} : \|\mathbf{x} - \boldsymbol{\alpha}\| < \rho\}$ $\boldsymbol{\alpha}$ being the root of the system of nonlinear equations. The iteration will converge if for any value of the starting iterate $\mathbf{x}_0 \in \Re$, the following condition is satisfied in some matrix norm:

   $$\|\mathbf{J}\| \le m < 1, \; \forall \; \mathbf{x} \in \Re$$

   A necessary condition for this to happen is that the spectral radius of J be less than for all $\mathbf{x} \in \Re$

4. Why is it necessary to obtain numerical estimates of the partial derivatives in multi-dimensions?

For finding the roots of a system of nonlinear equations it is necessary to obtain the components of the Jacobian matrix, which involves evaluating partial derivatives of the function $\boldsymbol{\varphi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots \phi_n(\mathbf{x}))$ This can be done by estimating the derivative by a difference quotient, e.g.

$$\frac{\partial \phi_i(\mathbf{x})}{\partial x_j} \approx \frac{\phi_i(\mathbf{x} + h_j \mathbf{e}_j) - \phi_i(\mathbf{x})}{h_j}$$

where e_j is the j^{th} unit vector and h_j is the j^{th} component of a n-dimensional parameter vector with components $h_j \neq 0$ for $j = 1,2\ldots.n$.