Lesson 13

1. Why are numerical algorithms with higher orders of accuracy not widely used?

   While there are many general methods for constructing iteration schemes with higher order of convergence, they entail increased computational cost. It can be shown that a one point iteration method with p^{th} order accuracy requires computation or p quantitites include *p-1* derivatives of the function at each step of the iteration i.e. it requires that $f(x_n), f'(x_n), \ldots, f^{(p-1)}(x_n)$ be evaluated in order for x_{n+1} to be found.

2. Why are round off errors crucial to the accuracy of a numerical solution for the root of a non-liner equation?

   Most estimates of the order of convergence of a numerical algorithm for finding the roots of a nonlinear equation assume that there is no round off error. However round off errors are crucial and for a sufficiently large number of iterations, the round off error is the dominant contribution to the error in the estimate of the root i.e. the error $|\bar{x}_{n+1} - \alpha|$ where $\bar{x}_{n+1}$ is the estimate to the root α after $n+1$ iterations is largely comprised of round off error. This is because the round off error remains constant with the number of iterations while the truncation error decreases with the number of iterations.

3. Why does the contribution of the round off to the error in the estimate of the root remain constant?

   In the $n+1^{th}$ step of the iteration, the round off error in the solution depends only on the round off error in the computation during the $n+1^{th}$ iteration. It does not depend on the round off errors in the previous iterations. This is because round off errors are self-correcting and do not accumulate. Because of this it is not necessary to compute with full accuracy (use double precision real storage for example) during the first few iterations.

4. What is the criterion for a function $f(x)$ to have multiple roots?

   The function $f(x)$ is said to possess roots of multiplicity q, say, at $x = \alpha$ if it is possible to define a function $g(x)$ such that $g(x) = (x-\alpha)^{-q} f(x)$ and $g(x)$ is bounded at $x = \alpha$ i.e. $0 < |g(\alpha)| < \infty$.

5. Does the order of convergence for a function with multiple roots the same as for a function with a simple root?

In general the order of convergence declines if the function has multiple roots as compared to a simple root. For example the Newton Raphson method is only linearly convergent at a multiple root while it is quadratically convergent at a simple root.

6. <u>How can the order of convergence for a function with multiple roots be made identical to the order of convergence for a function with simple roots?</u>

If the function $f(x)$ has q multiple roots at $x = \alpha$, and supposing that the function $f(x)$ is q times differentiable in a neighbourhood of α, then, if an auxiliary function $u(x)$ is defined such that $u(x) = \dfrac{f(x)}{f'(x)}$, $u(x)$ will have a simple root at α. Thus if instead of finding the roots of $f(x)$, the root of $u(x)$ is found, the numerical algorithms used for finding the root of $u(x)$ will have the desired rate of convergence.