Lesson 12

1. What is the order of convergence for the Newton Raphson method?

   For functions belonging to the function space C_2 i.e. for functions whose second derivatives are continuous, the Newton-Raphson method converges quadratically near the root .i.e $p = 2$. This order of convergence holds as long as round off errors in the calculations are small enough to be ignored.

2. What is the justification for the modified Newton method?

   The update formula for the Newton method is: $x_{n+1} = x_n + h_n, \quad h_n = -\dfrac{f(x_n)}{f'(x_n)}$

   Thus the slope of the function $f'(x_n)$ needs to be computed to the same relative accuracy as the function evaluation $f(x_n)$. Near the root $f(x_n)$ becomes smaller and smaller. Hence its relative accuracy is low as x_n approaches the root. Hence it may not be necessary to compute $f'(x_n)$ to unnecessary accuracy as $f(x_n)$ approaches the root. Hence it may not be necessary to compute $f'(x_n)$ at each iteration, as is done for the modified Newton method: the rate of convergence will only be slightly slower.

3. What is the secant method?

   The secant method is similar to the modified Newton method. However in this method, instead of the actual derivative, the secant approximation to the derivative is used. The secant method can be derived from Newton Raphson's method by approximating the derivative $f'(x_n)$ as:

   $$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \frac{f_n - f_{n-1}}{x_n - x_{n-1}}$$

4. What are the conditions for the secant method to converge and what is its order of convergence?

   The secant method converges when at the root α $f'(\alpha)$ is not equal to zero and the function $f(x)$ belongs to the class of C^2 functions. For the secant method $p = 1.618$. Since this is less than 2.0, the Secant method has less than quadratic convergence, unlike the Newton method.

5. What is the generalized *m*-point iteration method?

   In the generalized *m*-point iteration method, the $n+1$ th iterate is obtained from the function values and derivatives of $f(x)$ at 'm' points, i.e.

   $$x_{n+1} = \phi(x_n, x_{n-1}, \ldots x_{n-m})$$

   The function $\varphi(x)$ is known as the iteration function and is obtained by rewriting the function $f(x) = 0$ as $x = \varphi(x)$.

6. When will a generalized *m*-point iteration method converge?

   A generalized *m*-point iteration method will converge under the following conditions:

Suppose that $x = \phi(x)$ has a root α, and $\phi'(x)$ exists and satisfies the condition $|\phi'(x)| \leq m < 1$ in the interval $J = \{x: |x-\alpha| \leq \rho\}$. Then if the starting iterate x_0 belongs to *J*, all subsequent iterates $x_1, x_2 \ldots x_n$ will also lie in *J* and will converge to the root α if α is the only root of $x = \varphi(x)$ in *J*

Suppose that $x = \phi(x)$ has a root α, and $\phi'(x)$ exists and satisfies the condition $|\phi'(x)| \leq m < 1$ in the interval $J = \{x: |x-\alpha| \leq \rho\}$. Then, for all $x_0 \in J$: (a) $x_n \in J, n = 0,1,2....$ $(b) \lim_{n\to\infty} x_n = \alpha$

$(c)\alpha$ is the only root in J of $x = \varphi(x)$