Lesson 11

1. Why are numerical methods necessary for finding the roots of a nonlinear equation?

   The roots of nonlinear equations cannot generally be expressed in closed form. Thus numerical methods based either on linearization or successive approximation need to be used. These iterative methods start from an initial approximation to the root and produce a sequence which eventually converges to the desired value.

2. Since there are multiple roots to a nonlinear equation which root will the iterations or successive approximations converge to?

   In general, the root that the iterations converge to, depends on the starting approximation used for the iterations. Unless the starting value of the iteration is within a certain interval, say [a, b] that contains the root, the iteration may not converge to the desired root. Other algorithms may require an initial approximation that is close to the desired root in order to converge. These methods typically have the advantage that they converge more quickly.

3. Can a combination of iteration algorithms be more efficient in finding the root of a nonlinear equation than a single algorithm?

   Some methods do not converge very fast, but can still converge if the starting iterate is within a certain (somewhat large) interval of the root. Hence one can begin with a method that is relatively simple to use, which may not converge very fast, but is sure to converge if starting iterate is within a certain interval that contains the root. Then when the iterate gets close to the root, it may be desirable to switch to an algorithm which converges only for starting values close to the root, but converges much faster.

4. What is the method of bisection?

   The method of bisection is a relatively simple method to establish bounds for the root of a nonlinear function. It involves repeated evaluations of the function to successively narrow the interval for the root. It requires knowledge of an initial interval such that the function is positive at the lower bound of the interval and negative at the upper bound, or vice versa. The method will eventually converge to the root with an acceptable accuracy so long as the number of iterations is large. However convergence is very slow.

5. How is the rate of convergence of a nonlinear iterative method defined?

   If an iterative method generates a sequence $x_1, x_2 \ldots . x_n$ in a certain manner and the sequences converge to a value α, then we define the error for the n^{th} iterate x_n as $\varepsilon_n = x_n - \alpha$. Then if there exists a number p and a constant C with C not equal to zero, such that

$\lim_{n\to\infty} \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^p} = C$ then p is called the order of convergence of the sequence and C the asymptotic error constant. For $p = 1,2,3$ the convergence is said to be linear, quadratic, cubic etc.