Advanced Numerical Analysis for Chemical Engineerig
Quiz and Solution Key

1. Describe geometrically the subspaces of R^3 **spanned** by following sets

   (a) (1,0,1), (0,1,-1), (0,-2,0)

   Solution: *The above 3 vectors are linearly independent and span the entire 3 dimensional space, i.e.* R^3.

   (b) (-1,0,0), (5,0,-8), (0,0,4)

   Solution: *Only 2 out above 3 vectors are linearly independent and they span a 2 dimensional subspace in* R^3.

   (c) (2,-2,0), (-5,0,0), (0,3,0)

   Solution: *Only 2 out above 3 vectors are linearly independent and they span a 2 dimensional subspace in* R^3.

2. Determine whether following definition is valid as definitions for norm in $\mathbf{C}^{(2)}[a,b]$

   Solution: *Since we are dealing with real valued functions, for any function to qualify as inner product, it has to satisfy the following four axioms*

   A1. $\|\mathbf{x}\| \geq 0$ *for all* $x \in X$; $\|\mathbf{x}\| = 0$ *if and only if* $x = \overline{0}$ *(zero vector)*

   A2. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$*for each* $x, y \in X$. *(triangle inequality).*

   A3. $\|\alpha\mathbf{x}\| = |\alpha| \,.\, \|\mathbf{x}\|$ *for all scalars* α *and each* $x \in X$

   (a) $max\,|\mathbf{x}(t)| + \max|\mathbf{x}'(t)|$

   For any non-zero function $x(t)$, *axiom A1 is satisfied. Axiom A2 follows from the following inequality*

$$\begin{aligned}
\|\mathbf{x}(t)+\mathbf{y}(t)\| &= max\,|\mathbf{x}(t)+\mathbf{y}(t)| + \max|\mathbf{x}'(t)+\mathbf{y}'(t)| \\
&\leq [max\,|\mathbf{x}(t)| + max\,|\mathbf{y}(t)|] + [\max|\mathbf{x}'(t)| + \max|\mathbf{y}'(t)|] \\
&\leq [max\,|\mathbf{x}(t)| + \max|\mathbf{x}'(t)|] + [max\,|\mathbf{y}(t)| + \max|\mathbf{y}'(t)|] \\
&\leq \|\mathbf{x}(t)\| + \|\mathbf{y}(t)\|
\end{aligned}$$

   It is easy to show that axiom A3 is also satisfied for all scalars α. *Thus, given function defines a norm on* $C^{(2)}[a,b]$.

   (b) $|\mathbf{x}(a)| + \max|\mathbf{x}'(t)|$

For any non-zero function $x(t)$, axiom A1 is satisfied. Axiom A2 follows from the following inequality

$$\begin{aligned}
\|\mathbf{x}(t)+\mathbf{y}(t)\| &= |\mathbf{x}(a)+\mathbf{y}(a)| + \max|\mathbf{x}'(t)+\mathbf{y}'(t)| \\
&\leq [|\mathbf{x}(a)|+|\mathbf{y}(a)|] + [\max|\mathbf{x}'(t)| + \max|\mathbf{y}'(t)|] \\
&\leq [|\mathbf{x}(a)| + \max|\mathbf{x}'(t)|] + [|\mathbf{y}(a)| + \max|\mathbf{y}'(t)|] \\
&\leq \|\mathbf{x}(t)\| + \|\mathbf{y}(t)\|
\end{aligned}$$

Axiom A3 is also satisfied for any α as

$$\begin{aligned}
\|\alpha\mathbf{x}(t)\| &= |\alpha\mathbf{x}(a)| + \max|\alpha\mathbf{x}'(t)| \\
&= |\alpha|\,[|\mathbf{x}(a)| + \max|\mathbf{x}'(t)|] \\
&= |\alpha|\,.\,\|\mathbf{x}\|
\end{aligned}$$

(c) $|\mathbf{x}(a)| \max|\mathbf{x}(t)|$

Consider a non-zero function $x(t)$ in $C^{(2)}[a,b]$ such that $x(a)=0$ and $\max|\mathbf{x}(t)| \neq 0$. Then, axiom A1 is NOT satisfied for any vector $x(t)$ in $C^{(2)}[a,b]$ and the above function does not qualify to be a norm on $C^{(2)}[a,b]$.

3. Show that function $\langle \mathbf{x}, \mathbf{y} \rangle_W : R^n \times R^n \to R$ defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y}$$

defines an inner product on when W is a symmetric positive definite matrix.

Solution: *For $\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y}$ to qualify as inner product, it must satisfy the following axioms*

A1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$

A2. $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

A3. $\langle \lambda\mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$ *and* $\langle \mathbf{x}, \lambda\mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$

A4. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x = \overline{0}$.

We have, $\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y}$ and $\langle \mathbf{y}, \mathbf{x} \rangle_W = \mathbf{y}^T W \mathbf{x}$.Since W is symmetric, i.e. $W^T = W$, $\left[\mathbf{x}^T W \mathbf{y}\right]^T = \mathbf{y}^T W^T \mathbf{x} = \mathbf{y}^T W \mathbf{x}$. Thus, axiom A1 holds for any $\mathbf{x}, \mathbf{y} \in R^n$.

$\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle_W = (\mathbf{x}+\mathbf{y})^T W \mathbf{z} = \mathbf{x}^T W \mathbf{z} + \mathbf{x}^T W \mathbf{z} = \langle \mathbf{x}, \mathbf{z} \rangle_W + \langle \mathbf{y}, \mathbf{z} \rangle_W$. *Thus, axiom A2 holds for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^n$.*

$\langle \lambda\mathbf{x}, \mathbf{y} \rangle = (\lambda\mathbf{x})^T W \mathbf{y} = \lambda(\mathbf{x}^T W \mathbf{y}) = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$

$\langle \mathbf{x}, \lambda\mathbf{y} \rangle = \mathbf{x}^T W(\lambda\mathbf{y}) = \lambda(\mathbf{x}^T W \mathbf{y}) = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$. *Thus, axiom A3 holds for any $\mathbf{x}, \mathbf{y} \in R^n$.*

Since W is positive definite, it follows that $\langle \mathbf{x}, \mathbf{x} \rangle_W = \mathbf{x}^T W \mathbf{x} > 0$ *if* $x \neq \overline{\mathbf{0}}$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle_W = \mathbf{x}^T W \mathbf{x} = 0$ *if* $x = \overline{\mathbf{0}}$. *Thus, axiom A4 holds for any* $\mathbf{x} \in R^n$.

Since all four axioms are satisfied, $\langle \mathbf{y}, \mathbf{x} \rangle_W = y^T W x$ *is a valid definition of an inner product.*

4. The triangle inequality asserts that, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ belonging to an inner product space

$$\|\mathbf{x} + \mathbf{y}\|_2 \leq ||\mathbf{y}||_2 + ||\mathbf{x}||_2$$

After squaring both the sides and expanding,reduce this to Schwartz inequality. Under what condition Schwartz inequality becomes an equality?

Solution: *Squaring both the sides, we have*

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \leq [||\mathbf{y}||_2 + ||\mathbf{x}||_2]^2$$

$$\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + 2 \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2 ||\mathbf{x}||_2$$

$$||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2 \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2 ||\mathbf{x}||_2$$

Since, $||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 \geq 0$ *for any* $\mathbf{x}, \mathbf{y} \in X$, *the above inequality reduces to*

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2 ||\mathbf{x}||_2 \tag{1}$$

The triangle inequality also implies that

$$\|\mathbf{x} - \mathbf{y}\|_2^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \leq [||\mathbf{y}||_2 + ||\mathbf{x}||_2]^2$$

$$\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - 2 \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2 ||\mathbf{x}||_2$$

$$||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 - 2 \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 + 2||\mathbf{y}||_2 ||\mathbf{x}||_2$$

Since, $||\mathbf{y}||_2^2 + ||\mathbf{x}||_2^2 \geq 0$ *for any* $\mathbf{x}, \mathbf{y} \in X$, *the above inequality reduces to*

$$- \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2 ||\mathbf{x}||_2$$

i.e.

$$-||\mathbf{y}||_2 ||\mathbf{x}||_2 \leq \langle \mathbf{x}, \mathbf{y} \rangle \tag{2}$$

Combining inequalities (1) and (2), we arrive at the Cauchy-Schwartz inequality

$$-||\mathbf{y}||_2 ||\mathbf{x}||_2 \leq \langle \mathbf{x}, \mathbf{y} \rangle \leq ||\mathbf{y}||_2 ||\mathbf{x}||_2 \tag{3}$$

i.e.

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{y}||_2 ||\mathbf{x}||_2 \tag{4}$$

The Cauchy-Schwartz inequality reduces to equality when $\mathbf{y} = \alpha \mathbf{x}$.