

Advanced Process Control
Mid-semester Examination: Solution Key
(2 hours and 25 Marks)

1. Consider an ARMA process

$$v(k) = \alpha v(k-1) + e(k) + \beta e(k-1) \quad (1)$$

where $\{e(k)\}$ is a zero mean white noise process with variance λ^2 . It can be shown that stochastic process $\{v(k)\}$ has zero mean.

(a) Derive expressions for cross-covariance $r_{ve}(1) = E[v(k)e(k-1)]$ (2 marks)

Soln.:

$$\begin{aligned} r_{ve}(1) &= E[v(k)e(k-1)] \\ &= E[[\alpha v(k-1) + e(k) + \beta e(k-1)]e(k-1)] \\ &= \alpha E[v(k-1)e(k-1)] + \beta E[e(k-1)^2] \end{aligned}$$

$$\begin{aligned} E[v(k-1)e(k-1)] &= E[[\alpha v(k-2) + e(k-1) + \beta e(k-2)]e(k-1)] \\ &= E[e(k-1)^2] = \lambda^2 \\ &\Rightarrow r_{ve}(1) = (\alpha + \beta)\lambda^2 \end{aligned}$$

(b) Derive expressions for auto-covariance $r_v(1) = E[v(k)v(k-1)]$. (4 marks)

Soln: Approach 1:

$$\begin{aligned} r_v(1) &= E[v(k)v(k-1)] \\ &= E[[\alpha v(k-1) + e(k) + \beta e(k-1)][.]] \\ [. &= [\alpha v(k-2) + e(k-1) + \beta e(k-2)] \end{aligned}$$

Multiplying and retaining only non-zero terms

$$\begin{aligned} r_v(1) &= \alpha^2 E[v(k-1)v(k-2)] + \beta E[e(k-1)^2] \\ &\quad + \alpha E[v(k-1)e(k-1)] + \alpha\beta E[v(k-1)e(k-2)] \end{aligned}$$

Now, we have

$$\begin{aligned} r_v(1) &= E[v(k)v(k-1)] = E[v(k-1)v(k-2)] \\ E[v(k-1)e(k-1)] &= \lambda^2 \\ E[v(k-1)e(k-2)] &= r_{ve}(1) = (\alpha + \beta)\lambda^2 \end{aligned}$$

Thus, it follows that

$$(1 - \alpha^2)r_v(1) = (\alpha + \beta)\lambda^2 + \alpha\beta(\alpha + \beta)\lambda^2$$

$$\Rightarrow r_v(1) = \frac{\lambda^2}{1 - \alpha^2}(\alpha + \beta) [1 + \alpha\beta]$$

Approach 2:

$$\begin{aligned} r_v(1) &= E[v(k)v(k-1)] \\ &= E[[\alpha v(k-1) + e(k) + \beta e(k-1)]v(k-1)] \\ &= \alpha E[v(k-1)^2] + E[e(k)v(k-1)] + \beta E[e(k-1)v(k-1)] \\ &\quad E[e(k)v(k-1)] = 0 \end{aligned}$$

$$\begin{aligned} E[v(k)^2] &= E[\alpha v(k-1) + e(k) + \beta e(k-1)]^2 \\ &= \alpha^2 E[v(k-1)^2] + (1 + \beta^2)\lambda^2 + 2\alpha E[e(k)v(k-1)] \\ &\quad + 2(\alpha\beta)E[v(k-1)e(k-1)] + 2\beta E[e(k)e(k-1)] \\ &= \alpha^2 E[v(k-1)^2] + (1 + \beta^2 + 2\alpha\beta)\lambda^2 \end{aligned}$$

Since

$$r_v(0) = E[v(k)^2] = E[v(k-1)^2]$$

it follows that

$$E[v(k-1)^2] = r_v(0) = \frac{(1 + \beta^2 + 2\alpha\beta)\lambda^2}{1 - \alpha^2}$$

$$\begin{aligned} \Rightarrow r_v(1) &= \frac{\alpha(1 + \beta^2 + 2\alpha\beta)\lambda^2}{1 - \alpha^2} + \beta\lambda^2 \\ \Rightarrow r_v(1) &= \frac{\lambda^2}{1 - \alpha^2} [\alpha + \alpha\beta^2 + 2\alpha^2\beta + \beta - \alpha^2\beta] \\ \Rightarrow r_v(1) &= \frac{\lambda^2}{1 - \alpha^2} [\alpha + \beta + \alpha\beta^2 + \alpha^2\beta] \\ \Rightarrow r_v(1) &= \frac{\lambda^2}{1 - \alpha^2}(\alpha + \beta) [1 + \alpha\beta] \end{aligned}$$

2. Consider Box-Jenkin's model

$$y(k) = \frac{q^{-1} + 0.5q^{-2}}{(1 - 0.5q^{-1})(1 - 0.8q^{-1})}u(k) + \frac{1 + 0.5q^{-1}}{(1 - 0.8q^{-1})}e(k)$$

Derive one step prediction

$$\begin{aligned} \hat{y}(k|k-1) &= [H(q)]^{-1}G(q)u(k) + [1 - (H(q))^{-1}]y(k) \\ y(k) &= \hat{y}(k|k-1) + e(k) \end{aligned}$$

and express dynamics of $\hat{y}(k|k-1)$ as a time domain difference equation. (6 marks)

Soln.:

$$\begin{aligned} H(q)^{-1}G(q) &= \frac{(1 - 0.8q^{-1})}{(1 + 0.5q^{-1})} \frac{q^{-1}(1 + 0.5q^{-1})}{(1 - 0.5q^{-1})(1 - 0.8q^{-1})} \\ &= \frac{q^{-1}}{(1 - 0.5q^{-1})} \end{aligned}$$

$$1 - H(q)^{-1} = 1 - \frac{(1 - 0.8q^{-1})}{(1 + 0.5q^{-1})} = \frac{1.3q^{-1}}{(1 + 0.5q^{-1})}$$

$$\begin{aligned} \hat{y}(k|k-1) &= \frac{q^{-1}}{(1 - 0.5q^{-1})}u(k) + \frac{1.3q^{-1}}{(1 + 0.5q^{-1})}y(k) \\ &= \frac{q^{-1}(1 + 0.5q^{-1})u(k) + 1.3(1 - 0.5q^{-1})q^{-1}y(k)}{(1 - 0.5q^{-1})(1 + 0.5q^{-1})} \\ &= \frac{(q^{-1} + 0.5q^{-2})u(k) + (1.3q^{-1} - 0.65q^{-2})y(k)}{(1 - 0.25q^{-2})} \end{aligned}$$

$$\begin{aligned} (1 - 0.25q^{-2})\hat{y}(k|k-1) &= (q^{-1} + 0.5q^{-2})u(k) + (1.3q^{-1} - 0.65q^{-2})y(k) \\ \hat{y}(k|k-1) &= 0.25\hat{y}(k-2|k-3) + u(k-1) + 0.5u(k-2) \\ &\quad + 1.3y(k-1) - 0.65y(k-2) \end{aligned}$$

3. Consider a coupled tank system in which dynamics of levels in the two tanks is governed by

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

where x denotes perturbations in level and u denotes perturbations in inlet flow.

- (a) It is desired to control this system (at the setpoint equal to the origin) using a feedback control law of the form

$$u = - \begin{bmatrix} \alpha & \beta \end{bmatrix} \mathbf{x}$$

Determine the state space model (differential equation) that governs the closed loop dynamics in terms of unknowns (α, β) . (2 marks)

Soln.: *The closed loop equation is given by*

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} \mathbf{x} \\ &= \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} -\alpha & -\beta \\ \alpha & \beta \end{bmatrix} \mathbf{x} \\ &= \begin{bmatrix} -3 + \alpha & 2 + \beta \\ -\alpha & -1 - \beta \end{bmatrix} \mathbf{x} \end{aligned}$$

- (b) Determine, if it exists, controller gains $[\alpha \ \beta]$ such that the state transition matrix for the closed loop system has eigenvalues at the roots of the following quadratic equation (4 marks)

$$\lambda^2 + 11\lambda + 30 = 0$$

Soln.: *Eigenvalues of the closed loop state transfer matrix are at the root of*

$$\begin{aligned} \det \left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 + \alpha & 2 + \beta \\ -\alpha & -1 - \beta \end{bmatrix} \right] &= 0 \\ \det \begin{bmatrix} \lambda + 3 - \alpha & -2 - \beta \\ \alpha & \lambda + 1 + \beta \end{bmatrix} &= 0 \\ (\lambda + 3 - \alpha)(\lambda + 1 + \beta) + \alpha(2 + \beta) &= 0 \\ \lambda^2 + (4 - \alpha + \beta)\lambda + (3 - \alpha)(1 + \beta) + \alpha(2 + \beta) &= 0 \\ \lambda^2 + (4 - \alpha + \beta)\lambda + (3 - \alpha)(1 + \beta) + \alpha(2 + \beta) &= 0 \\ \lambda^2 + (4 - \alpha + \beta)\lambda + (3 + 3\beta + \alpha) &= 0 \end{aligned}$$

For this characteristic equations to have eigen values identical to that of

$$\lambda^2 + 11\lambda + 30 = 0$$

we require

$$\begin{aligned} (4 - \alpha + \beta) &= 11 \\ -\alpha + \beta &= 7 \end{aligned} \tag{2}$$

$$\begin{aligned} (3 + 3\beta + \alpha) &= 30 \\ 3\beta + \alpha &= 27 \end{aligned} \tag{3}$$

Adding equation (1) with (2), we have

$$\begin{aligned} 4\beta &= 34 \Rightarrow \beta = \frac{17}{2} \\ \alpha &= \beta - 7 = \frac{3}{2} \end{aligned}$$

4. Consider a **continuous time** linear perturbation model

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} -2 & 1/2 & 1/2 \\ 1 & -3/2 & -1/2 \\ 1 & 1/2 & -5/2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \mathbf{u}$$

Eigenvalues of matrix \mathbf{A} are -1, -2 and -3 and the continuous time system is asymptotically stable. Suppose discretization of the continuous time system is carried out using the Euler's method i.e. $[\Phi]_{Euler} = \mathbf{I} + T\mathbf{A}$. Then, find the range of sampling time T for which the discrete time model will retain the stability characteristics of the continuous time system. (7 marks)

Hint: If matrix A is diagonalizable, can you relate eigenvalues of \mathbf{A} with eigenvalues of $[\Phi]_{Euler}$?

Soln.: Since eigen values of A matrix are distinct, eigen vectors of A are linearly independent and matrix A is diagonalizable.

$$\begin{aligned}\mathbf{A} &= \Psi\Lambda\Psi^{-1} \\ \Rightarrow \Phi_{Euler} &= \mathbf{I} + T\Psi\Lambda\Psi^{-1} = \Psi[\mathbf{I} + T\Lambda]\Psi^{-1}\end{aligned}$$

Thus, eigen values of Φ_{Euler} are diagonal elements of matrix $[\mathbf{I} + T\Lambda]$. For the particular system under consideration

$$\begin{aligned}\Lambda &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \\ \mathbf{I} + T\Lambda &= \begin{bmatrix} 1 - T & 0 & 0 \\ 0 & 1 - 2T & 0 \\ 0 & 0 & 1 - 3T \end{bmatrix}\end{aligned}$$

Alternatively, let \mathbf{v} represent eigen vector of \mathbf{A} corresponding to eigen value λ . Then, we have

$$\Phi_{Euler}\mathbf{v} = (\mathbf{I} + T\mathbf{A})\mathbf{v} = \mathbf{v} + T\lambda\mathbf{v} = (1 + T\lambda)\mathbf{v}$$

i.e. $(1 + T\lambda)$ is eigenvalue of Φ_{Euler} . Thus, eigenvalues of Φ_{Euler} are $1 - T, 1 - 2T$ and $1 - 3T$.

Alternatively,

$$(\mathbf{I} + T\mathbf{A}) - \lambda\mathbf{I} = \begin{bmatrix} 1 - 2T - \lambda & T/2 & T/2 \\ T & 1 + (-3/2)T - \lambda & -T/2 \\ T & T/2 & 1 + (-5/2)T - \lambda \end{bmatrix}$$

Setting

$$\det [(\mathbf{I} + T\mathbf{A}) - \lambda\mathbf{I}] = 0$$

and rearranging, we have

$$\begin{aligned}(\lambda - 1)^3 + 6T(\lambda - 1)^2 + 11(\lambda - 1)T^2 + 6T^3 &= 0 \\ (\lambda - 1 + T)(\lambda - 1 + 2T)(\lambda - 1 + 3T) &= 0\end{aligned}$$

Thus, eigenvalues of Φ_{Euler} are $1 - T$, $1 - 2T$ and $1 - 3T$.

Asymptotic stability of the discretized system is guaranteed if and only if T is chosen such that

$$|1 - T| < 1, |1 - 2T| < 1 \text{ and } |1 - 3T| < 1$$

$$\begin{aligned} |1 - T| < 1 &\Rightarrow -1 < 1 - T < 1 \\ &\Rightarrow 0 < T < 2 \end{aligned} \tag{4}$$

$$\begin{aligned} |1 - 2T| < 1 &\Rightarrow -1 < 1 - 2T < 1 \\ &\Rightarrow 0 < T < 1 \end{aligned} \tag{5}$$

$$\begin{aligned} |1 - 3T| < 1 &\Rightarrow -1 < 1 - 3T < 1 \\ &\Rightarrow 0 < T < 2/3 \end{aligned} \tag{6}$$

If T is chosen such that inequality (5) is satisfied, then it follows that inequalities (3) and (4) are also satisfied. Thus, the discrete time system will retain the stability characteristics of the continuous time system if and only if T is chosen such that $0 < T < 2/3$.