Advanced Process Control
(Time: 3 hours)
End Semester Examination (40 Marks)
Instructions. *Closed Book* and *Closed Notes* examination.

1. **Pole placement controller and Kalman predictor design:** Consider the following difference equation

$$
\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -2 \\ 2 \end{bmatrix} u(k) + \begin{bmatrix} 2 \\ 2 \end{bmatrix} w(k) \\
y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k) + v(k)
\end{aligned}
$$

$$w(k) \sim N(0, 1/4) \;\; \text{and} \;\; v(k) \sim N(0, 1/4)$$

   (a) It is desired to develop a state feedback control law of the form

$$u(k) = -G\mathbf{x}(k)$$

   Find matrix G such that the poles of $(\Phi - \Gamma G)$ are placed at $q = 0.25 \pm 0.25j$. *(6 marks)*

   (b) For the dynamic system described above, **set up equations** for designing the steady state Kalman predictor of the form

$$\widehat{\mathbf{x}}(k+1) = \Phi\widehat{\mathbf{x}}(k|) + \mathbf{\Gamma u}(k) + \mathbf{L}\left[y(k) - \mathbf{C}\widehat{\mathbf{x}}(k)\right]$$

   where $\mathbf{L}$ represents steady state Kalman gain. *(6 marks).*
   Note: *Algebraic Riccati Equations are as follows*

$$
\begin{aligned}
\mathbf{P} &= \Phi\mathbf{P}\Phi^T + \mathbf{Q} - \mathbf{LCP}\Phi^T \\
\mathbf{L} &= \Phi\mathbf{PC}^T\left(\mathbf{R} + \mathbf{CPC}^T\right)^{-1}
\end{aligned}
$$

   where matrix $\mathbf{P}$ *is of the form*

$$\mathbf{P} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

   You are NOT expected to solve the resulting equations. Only state the equations in terms of unknowns (a,b,c).

2. **Stability, State Estimation and State Realization**

   (a) Consider the following difference equation representing dynamic behavior of a satellite.

$$
\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} (T^2)/2 \\ T \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)
\end{aligned}
$$

   where T represents the sampling interval. Is this system observable and reachable for any choice of the sampling time T ? *(4 marks)*

(b) Consider a discrete time system

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.5 & 1 \\ -0.5 & 0.5 \end{bmatrix} \mathbf{x}(k)$$

Determine whether the following function below qualifies to be a Lyapunov function for this system *(5 marks)*

$$V[\mathbf{x}(k)] = [x_2(k)]^2 + [x_1(k) + 2x_2(k)]^2$$

(c) For the following model identified from input-output data

$$y(k) = \frac{q-1}{(q-0.5)(q-0.4)} u_1(k) + \frac{2q+1}{(q-0.4)} u_2(k) + \frac{(q+0.6)}{(q-0.5)} e(k)$$

where $\{e(k)\}$ is a zero mean white noise sequence with variance 0., derive state realization

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi x}(k) + \mathbf{\Gamma} u(k) + \mathbf{L} e(k) \\ \mathbf{y}(k) &= \mathbf{Cx}(k) + \mathbf{D} u(k) + e(k) \end{aligned}$$

in the **observable canonical form** . *(7 marks)*

3. **Predictive Control and Model Matching Control Design**

(a) It is desired to develop a conventional MPC type predictive controller using a model of form

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}(k) + e(k) \end{aligned}$$

and *open loop* observer

$$\begin{aligned} \widehat{\mathbf{x}}(k) &= \Phi \widehat{\mathbf{x}}(k-1) + \Gamma u(k-1) \\ e(k) &= y(k) - C\widehat{\mathbf{x}}(k) \end{aligned}$$

using two step ahead predictions and control horizon equal to two.

i. At sampling instant k, derive two step ahead future predictions using the model with the initial condition as $\widehat{\mathbf{x}}(k)$ and arrange the predictions in following matrix equation

$$\mathbf{Y}(k) = \begin{bmatrix} \widehat{\mathbf{y}}(k+1|k) \\ \widehat{\mathbf{y}}(k+2|k) \end{bmatrix} = \mathbf{S}\widehat{\mathbf{x}}(k) + \mathbf{GU}(k) + \mathbf{L}e(k)$$

where matrices **S**, **G** are **L** are obtained using given model matrices (Φ,Γ, C) and

$$\mathbf{U}(k) = \begin{bmatrix} u(k|k) \\ u(k+1|k) \end{bmatrix}$$

(3 marks)

ii. Let

$$\mathbf{R}(k) = \begin{bmatrix} \mathbf{r}(k+1) \\ \mathbf{r}(k+2) \end{bmatrix}$$

define vector of future setpoints. Then, a control law that minimizes two norm of the future prediction error

$$\mathbf{E}(k) = \mathbf{R}(k) - \mathbf{Y}(k)$$

is given by setting future error $\mathbf{E}(k) = \overline{\mathbf{0}}$. Thus, find vector $\mathbf{U}(k)$ that will meet constraint $\mathbf{R}(k) = \mathbf{Y}(k)$ at each sampling instant. (*3 marks*).

(b) Consider process governed by

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \mathbf{\Gamma u}(k) \quad (1)$$

$$\mathbf{y}(k) = \mathbf{Cx}(k) \quad (2)$$

Further assume that number of manipulated inputs equals the number of controlled outputs. It is desired to arrive at a state feedback control law such that the output dynamics is governed by

$$\mathbf{y}(k+1) = \mathbf{Ay}(k) + (\mathbf{I} - \mathbf{A})\mathbf{r}(k) \quad (3)$$

where $\mathbf{r}(k)$ represents the setpoint.

i. Find $\mathbf{u}(k)$ as a function of $\mathbf{x}(k)$ such that the dynamics of $\mathbf{y}(k)$ generated by combining equations (1)-(2) exactly matches dynamics of $\mathbf{y}(k)$ given by (3). Does it yield a state feedback control law? (*4 marks*)

ii. Is there any condition that needs to be satisfied by some of the model matrices for such control law to exist? (*3 marks*)
Justify your answers in each case.