Chapter 12

Feedback Control

Feedback control allows a system dynamic response to be modified without changing any system components.

Below, we show an open-loop system (a system without feedback) and a closed-loop system (a system with feedback).

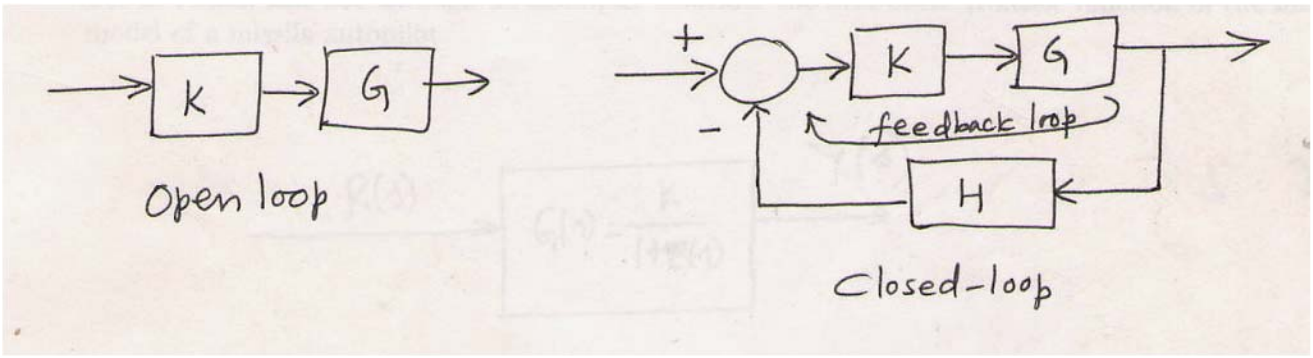


Figure 12.1: *Open loop and closed-loop system*

The H block is the transfer function that represents the dynamics of the feedback loop.

12.1 Characteristics of Feedback Systems

Advantages

1. Faster time response.
2. Better stability.

3. Less sensitive to noise.
4. Less sensitive to system parameter variations.

Disadvantages

1. May have a tendency to oscillate.
2. Cost becomes higher.

12.2 Objectives of Feedback Control

1. Speed (Rise time)
2. Accuracy (settling time and steady-state error)
3. Stability (Overshoot)
4. Robustness (will be treated in frequency response methods)

12.3 Types of Feedback Control

1. Proportional control (P-control)
2. Proportional-Integral or Integral control (PI-control)
3. Proportional-Integral-Derivative control (PID-control)

12.4 What Motivates Feedback Control?

Let us reason this out through an example. Consider the first order transfer function of the linearized model of a missile autopilot.

$$G(s) = \frac{k}{1 + \tau s}$$

where τ is the autopilot time constant. Now, let us apply a unit step input $R(s) = 1/s$.

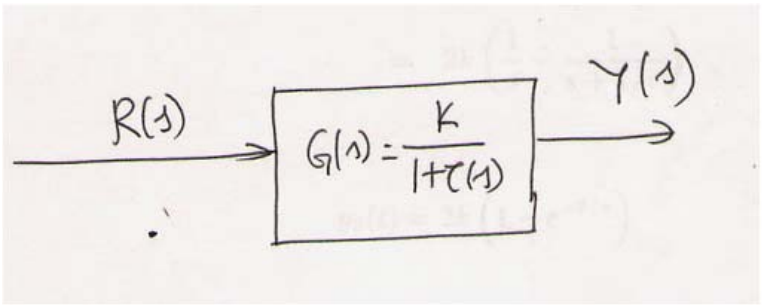


Figure 12.2: *A first order system*

Then the output is,

$$\begin{aligned} Y_1(s) &= G(s)R(s) = \frac{k}{1+\tau s} \cdot \frac{1}{s} \\ &= \frac{k}{\tau} \cdot \frac{1}{s+1/\tau} \cdot \frac{1}{s} \\ &= k\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right) \end{aligned}$$

So,

$$y_1(t) = k\left(1 - e^{-t/\tau}\right)$$

The response is fairly sluggish. What can we do to make the response faster without actually changing the autopilot?

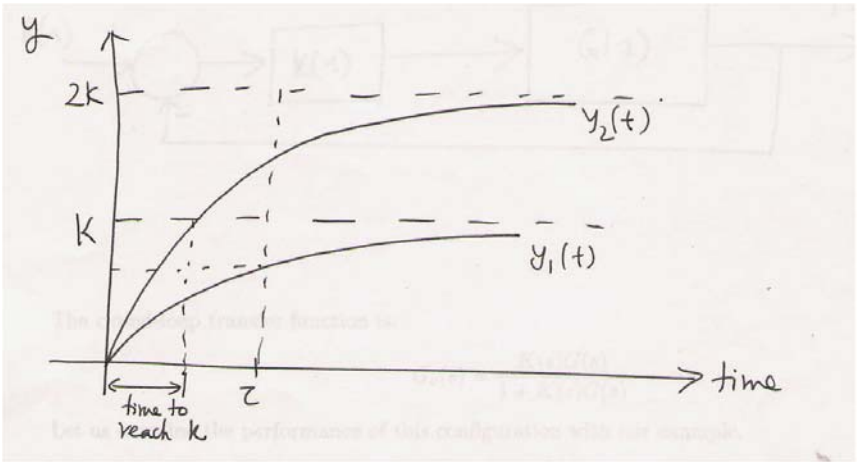


Figure 12.3: *What happens when we double the input?*

Let us double the input, that is, $R(s) = 2/s$. Then,

$$Y_2(s) = G(s)R(s) = \frac{k}{1+\tau s} \cdot \frac{2}{s}$$

$$= \frac{2k}{\tau} \cdot \frac{1}{s+1/\tau} \cdot \frac{1}{s}$$
$$= 2k\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right)$$

So,

$$y_2(t) = 2k\left(1 - e^{-t/\tau}\right)$$

There is actually no difference between the two time responses. Both have the same sluggish response as they have the same type of damping. But, look at the figure above. You can see that $y_2(t)$ attains the level k much earlier than $y_1(t)$ does. This gives rise to the following idea:

Why don't we apply a high input initially so that the system responds quickly and then decrease the input later?

In other words, instead of driving $G(s)$ with a step input, drive it with an input that is high initially and then drops down gradually. One way to do this would be to use the difference between the input step signal and the output and then drive the plant $G(s)$ with a high gain on this difference. This difference is usually known as the error signal. Note that initially this error is high (actually, it is the same as the reference input) and then it gradually decreases with time as the output attains values close to the reference input. The following block diagram achieves this and is called the feedback configuration. It is also called the closed-loop configuration.

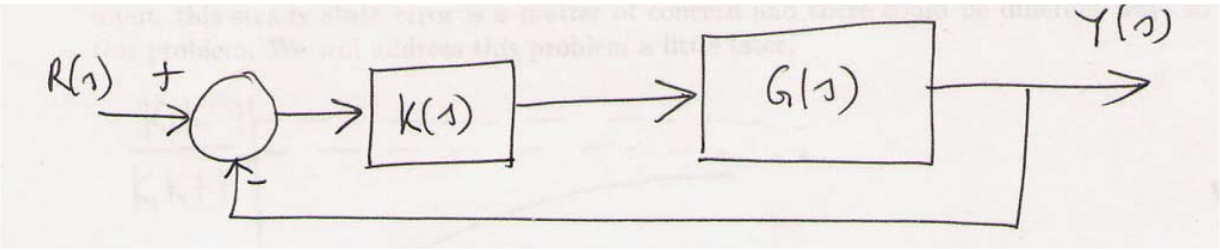


Figure 12.4: *A closed loop configuration*

The closed-loop transfer function is,

$$G_c(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

Let us examine the performance of this configuration with our example.

let, $K(s) = k_1$, which is a pure DC gain.

Since,

$$G(s) = \frac{k}{1+\tau s}$$

we have

$$G_c(s) = \frac{\frac{k_1 k}{1+\tau s}}{1+\frac{k_1 k}{1+\tau s}} = \frac{k_1 k}{1+\tau s + k_1 k} = \frac{k_1 k}{\tau\left(s+\frac{k_1 k+1}{\tau}\right)}$$

Now, apply a unit step input $R(s) = 1/s$.

$$\begin{aligned} Y(s) &= \frac{1}{s} \cdot \frac{k_1 k}{\tau\left(s+\frac{k_1 k+1}{\tau}\right)} \\ &= \frac{k_1 k}{k_1 k+1}\left[\frac{1}{s} - \frac{1}{s+\frac{k_1 k+1}{\tau}}\right] \end{aligned}$$

So,

$$\begin{aligned} y(t) &= \frac{k_1 k}{k_1 k+1}\left[1 - e^{-\left(\frac{k_1 k+1}{\tau}\right)t}\right] \\ &= \frac{k_1 k}{k_1 k+1}\left[1 - e^{-\frac{t}{\tau/(k_1 k+1)}}\right] \\ &= \frac{k_1 k}{k_1 k+1}\left[1 - e^{-\frac{t}{\tau_c}}\right] \end{aligned}$$

where,

$$\tau_c = \frac{\tau}{k_1 k+1}$$

is the time constant of the closed-loop system. One can observe that by selecting the value of k_1 we can reduce the time constant of the system.

But there is a problem here. The steady state value of the output is

$$\frac{k_1 k}{k_1 k+1}$$

which is less than the reference input 1 to the system. Since we want the system to follow the reference input, this steady state error is a matter of concern and there could be different ways to take care of this problem. We will address this problem a little later.

In the above example, the feedback control is just a simple gain, but it served our purpose of decreasing the time constant quite effectively. This configuration is called *proportional control* or *P-control*.

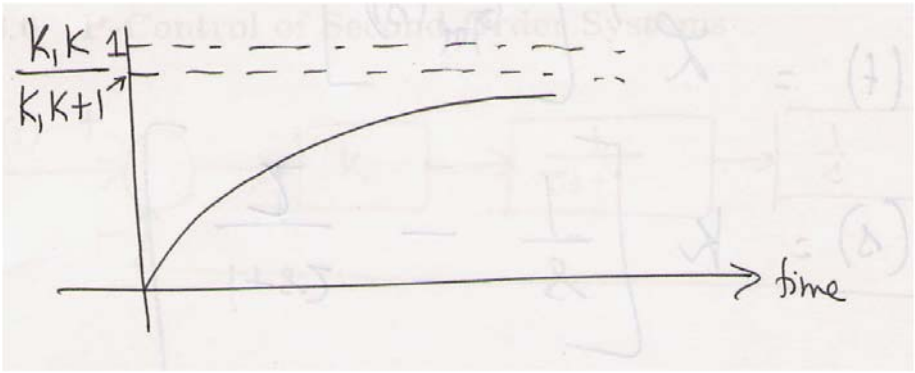


Figure 12.5: *An unwanted steady state error*

12.5 P-Control of First Order Systems

Let,

$$G(s) = \frac{1}{1+\tau s}$$

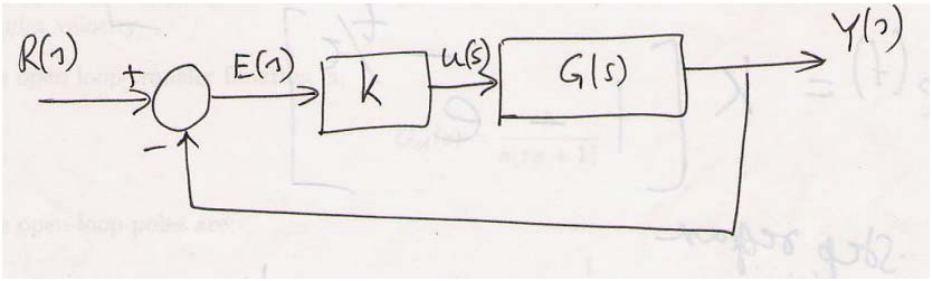


Figure 12.6: *P-control configuration*

Then,

$$G_c(s) = \frac{k}{k+1} \cdot \frac{1}{\frac{\tau}{k+1}s+1} = \bar{k}\frac{1}{1+\bar{\tau}s}$$

So,

	Open Loop	Closed Loop
Gain	1	$\bar{k} = \frac{k}{k+1}$ (decreases slightly)
Time Constant	τ	$\bar{\tau} = \frac{\tau}{k+1}$
Rise Time (T_r)	$\cong 2.2\tau$	$\cong 2.2\frac{\tau}{k+1}$
Settling Time (T_s)	$\cong 3.9\tau$	$\cong 3.9\frac{\tau}{k+1}$

So, there is considerable improvement in terms of rise time and settling time even though the DC gain reduces slightly giving rise to a non-zero steady state error.

12.6 P-Control of Second Order Systems

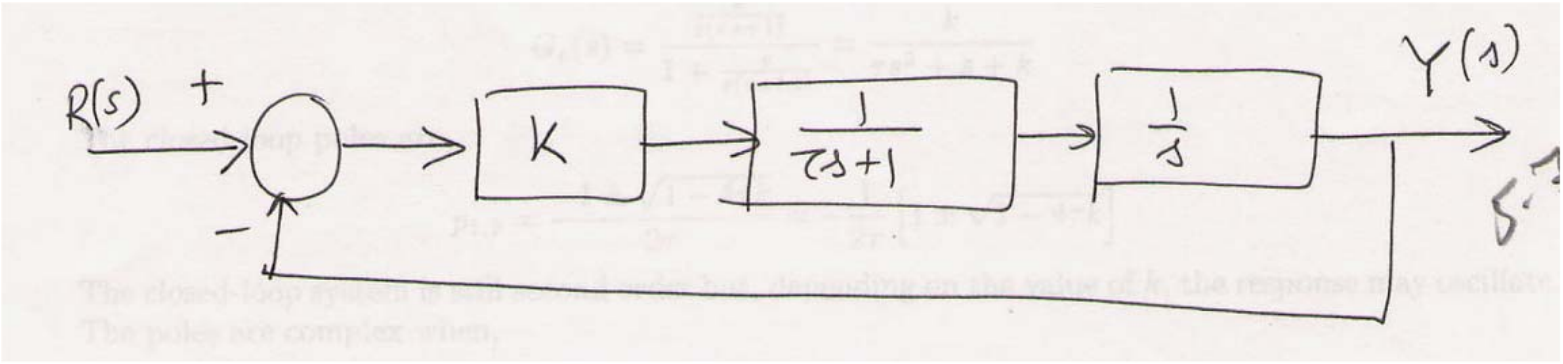


Figure 12.7: *P-control configuration for second order system*

Let us begin with an example shown in the above figure. The figure represents a missile autopilot with the missile lateral acceleration as its output. This lateral acceleration is integrated to obtain the missile angular velocity.

The open loop transfer function is,

$$G_{ol}(s) = \frac{k}{s(\tau s + 1)}$$

The open-loop poles are:

$$p_1 = 0; \quad p_2 = -\frac{1}{\tau}$$

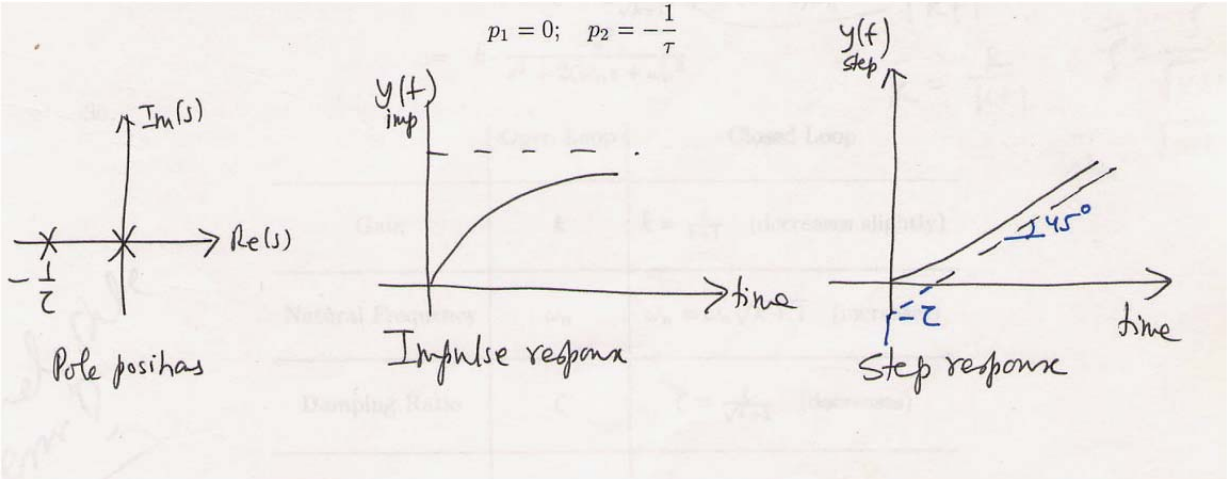


Figure 12.8: *Pole positions and responses*

The impulse response and the unit step response of the open-loop system are given by,

$$\begin{aligned} y_{impulse} &= k\left(1 - e^{-t/\tau}\right) \\ y_{step} &= k\left(-\tau + t + \tau e^{-t/\tau}\right) \end{aligned}$$

and are shown in the above figure.

Note that the open loop response is not oscillatory as there are no complex conjugate poles.

The closed-loop transfer function is,

$$G_c(s) = \frac{\frac{k}{s(\tau s+1)}}{1 + \frac{k}{s(\tau s+1)}} = \frac{k}{\tau s^2 + s + k}$$

The closed-loop poles are,

$$p_{1,2} = \frac{-1 \pm \sqrt{1 - 4\tau k}}{2\tau} = -\frac{1}{2\tau}\left[1 \pm \sqrt{1 - 4\tau k}\right]$$

The closed-loop system is still second order but, depending on the value of k, the response may oscillate. The poles are complex when,

$$1 - 4k\tau < 0 \Leftrightarrow k > \frac{1}{4\tau}$$

This shows that the choice of k is very crucial to the kind of time response we are looking for.

Now, consider the effect of P-control of a general second order system. The open-loop system is,

$$G_{ol}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The closed-loop system with P-control gain of k is,

$$\begin{aligned} G_c(s) &= \frac{kG_{ol}(s)}{1 + kG_{ol}(s)} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + (k+1)\omega_n^2} \\ &= \frac{k}{k+1} \cdot \frac{(k+1)\omega_n^2}{s^2 + 2\frac{\zeta}{\sqrt{k+1}}\sqrt{k+1}\omega_n s + (k+1)\omega_n^2} \\ &= \bar{k} \cdot \frac{\bar{\omega_n}^2}{s^2 + 2\bar{\zeta}\bar{\omega_n} s + \bar{\omega_n}^2} \end{aligned}$$

where,

$$\bar{k} = \frac{k}{k+1} \qquad \bar{\zeta} = \frac{\zeta}{\sqrt{k+1}} \qquad \bar{\omega}_n = \sqrt{k+1}\omega_n$$

So,

	Open Loop	Closed Loop
Gain	1	$\bar{k} = \frac{k}{k+1}$ (decreases slightly)
Natural Frequency	ω_n	$\bar{\omega_n} = \omega_n\sqrt{k+1}$ (increases)
Damping Ratio	ζ	$\bar{\zeta} = \frac{\zeta}{\sqrt{k+1}}$ (decreases)
Rise Time (T_r)	$\cong \frac{\pi}{2\omega_n}$	$\cong \frac{\pi}{2\sqrt{k+1}\ \omega_n}$ (decreases)
Overshoot (M_p)	$\cong 1 - \frac{\zeta}{0.6}$	$\cong 1 - \frac{\zeta}{0.6\sqrt{k+1}}$ (increases)
Settling Time (T_s)	$\cong \frac{4}{\zeta\omega_n}$	$\cong \frac{4}{\zeta\omega_n}$ (no change)

However, a word of caution here is necessary. We have used only the approximate relationships, and in some cases they may not be exactly valid. You may need to check with the exact expressions.